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Weak hyperbolicity of cube complexes and quasi-arboreal groups

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Abstract

We examine a graph Γ encoding the intersection of hyperplane carriers in a CAT(0) cube complex $\tilde{X}$. The main result is that Γ is quasi-isometric to a tree. This implies that a group G acting properly and cocompactly on $\tilde{X}$ is weakly hyperbolic relative to the hyperplane stabilizers. Using Wright's recent result on the asymptotic dimension of CAT(0) cube complexes, we give a generalization of a theorem of Bell and Dranishnikov on the finite asymptotic dimension of graphs of asymptotically finite-dimensional groups. Finally, we apply contact graph techniques to prove a cubical version of the flat plane theorem stated in terms of complete bipartite subgraphs of Γ.

1. *Introduction*

The set $\mathcal{W}$ of hyperplanes in a CAT(0) cube complex $\tilde{X}$ admits of a *crossing* relation and, more generally, a *contact* relation: distinct hyperplanes $W_1, W_2 \in \mathcal{W}$ *contact* if they have dual 1-cubes c_1, c_2 that have a common 0-cube. In particular, W_1 and W_2 contact if they *cross*, which happens when c_1 and c_2 form the corner of a 2-cube. The contact relation is encoded in a *contact graph* Γ, whose vertex set is $\mathcal{W}$ and whose edges correspond to contacting pairs of hyperplanes. The crossing relation gives a *crossing graph* $\Delta \subseteq \Gamma$, with the same vertex set, whose edges correspond to crossing pairs of hyperplanes.

The goal of this paper is to describe some properties of the contact graph and illustrate some uses of the contact graph and disk diagrams in studying CAT(0) cube complexes and cubulated groups. A geometric advantage of examining the contact graph is that, unlike the crossing graph, it is always connected. Moreover, in Section 4, we prove the following theorem.

Theorem (Theorem 4.1). *The contact graph Γ associated to a* CAT(0) *cube complex $\tilde{X}$ is quasi-isometric to a tree.*

Hence, cubulating a group entails construction of an action on a quasi-tree. Theorem 4.1 can be deduced from Manning's 'bottleneck' condition characterizing quasi-trees [**27**]; we also give a more constructive proof using disk diagram techniques, by constructing a *graded root tree* $\mathcal{T}$ and exhibiting a quasi-isometry $\Gamma \to \mathcal{T}$ which grades the hyperplanes by the distances of their images to a specified base vertex in $\mathcal{T}$.

Farb introduced the notion of *weak hyperbolicity* of a group G relative to a collection of subgroups $\{P\}$ to mean that the metric space obtained from the Cayley graph of G by 'coning off' each P-coset is δ-hyperbolic. In analogy, we define G to be 'weakly free' or *quasi-arboreal* relative to subgroups $\{P\}$ if the coned-off Cayley graph is quasi-isometric to a tree. We examine this acute form of weak hyperbolicity in Section 5, where we obtain the following consequence of Theorem 4.1.

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Corollary (Corollary 5.4). *Let G act properly and cocompactly on the* CAT(0) *cube complex $\tilde{X}$. Then G is quasi-arboreal relative to the set of hyperplane stabilizers.*

Section 6 discusses the asymptotic dimension of cubulated groups. Recently, in [**37**], Wright proved a beautiful theorem stating that the asymptotic dimension of a CAT(0) cube complex is bounded above by its dimension, and observed that this implies that groups acting properly on CAT(0) cube complexes have finite asymptotic dimension. On the other hand, Bell and Dranishnikov [**2**] showed that a finite graph of asymptotically finite-dimensional groups has finite asymptotic dimension. Using Wright's theorem on asymptotic dimension of cube complexes and the *Hurewicz-type theorem* of Bell and Dranishnikov [**3**], we obtain the following improved statement.

Corollary (Corollary 6.3). *Let G be a finitely generated group acting on the locally finite* CAT(0) *cube complex $\tilde{X}$, with $\dim \tilde{X} = D < \infty$. Suppose that there exists $n \in \mathbb{N}$ such that for each 0-cube x, the stabilizer G_x satisfies $\operatorname{asdim} G_x \leqslant n$. Then $\operatorname{asdim} G \leqslant n + D$.*

Section 7 discusses the relationship between Gromov–hyperbolicity of CAT(0) cube complexes and complete bipartite subgraphs of the associated crossing graph. The primary aim is the following theorem.

Theorem (Theorem 7.3). *Let G be a group acting properly and cocompactly on the* CAT(0) *cube complex $\tilde{X}$. Then exactly one of the following holds*:

(1) *G is word-hyperbolic*;
(2) *the crossing graph Δ of $\tilde{X}$ contains a complete bipartite graph $K_{\infty,\infty}$.*

Theorem 7.3 is a cubical version of the flat plane theorem (see, for example, [**6**]). The theorem is proved by constructing the $K_{\infty,\infty}$ from a sequence of arbitrarily large finite complete bipartite graphs, much as one constructs a plane as a limit of arbitrarily large disks in the proof of the flat plane theorem.

Sections 2 and 3 contain preliminary material: Section 2 summarizes the relevant properties of CAT(0) cube complexes and surveys techniques for manipulating disk diagrams in nonpositively curved cube complexes. These techniques appear in unpublished lecture notes of Casson, although not strictly in the context of CAT(0) cube complexes. They were developed further by Sageev in his thesis, and are described extensively by Wise in recent work. Moreover, Chepoi [**12**] has used disk diagram techniques in his proof that CAT(0) cube complexes are median spaces. Section 3 describes spheres in contact graphs.

2. *Preliminaries*

The following notions and notation are used throughout.

2.1. CAT(0) *cube complexes*

Definition 2.1 (Cube complex). For $0 \leqslant n < \infty$, an *n-cube* is a copy of the Euclidean cube $[-\frac{1}{2}, +\frac{1}{2}]^n$. A *$d$-dimensional face* of the n-cube c is a subspace obtained by restricting $n-d$ coordinates to $\pm\frac{1}{2}$. A *cube complex* X is a CW-complex whose n-dimensional cells are n-cubes, such that the attaching map of each cube c restricts to a combinatorial isometry on each face of c, mapping the face to a cube of X.

The *link* of a 0-cube v in a cube complex X is the complex made of simplices whose n-simplices correspond to the $(n+1)$-cubes that have a corner at v, with simplices attached along their faces according to the attaching of the corresponding cubes.

A simplicial complex S is a *flag complex* if each family of $n+1$ pairwise-adjacent 0-simplices in S spans an n-simplex, for each $n \geqslant 0$. A cube complex X is *nonpositively curved* if the link of v is a flag complex for every 0-cube v of X. A simply connected nonpositively curved cube complex $\tilde{X}$ is called a CAT(0) *cube complex*.

The term 'CAT(0) cube complex' is an artifact of the result of Gromov stating that a simply connected finite-dimensional cube complex satisfying the nonpositive curvature condition of Definition 2.1 admits of a piecewise-Euclidean CAT(0) metric [**18**]. This also follows from more general results of Bridson, in the finite-dimensional case [**5**], and was extended by Leary to infinite-dimensional cube complexes [**26**]. When we mention the CAT(0) metric on a CAT(0) cube complex $\tilde{X}$, we are referring to this metric. However, as discussed below, we shall usually use the more natural combinatorial metric on $\tilde{X}^{(1)}$.

Definition 2.2 (Hyperplane). A *midcube* of an n-cube c is an $(n-1)$-cube in c obtained by restricting exactly one coordinate to 0. A *hyperplane* W in the CAT(0) cube complex $\tilde{X}$ is a connected union of midcubes of cubes in $\tilde{X}$ such that, for each finite-dimensional cube c of $\tilde{X}$, either $W \cap c = \emptyset$ or $W \cap c$ is a single midcube of c. The *carrier* $N(W)$ of W is the union of all closed cubes c such that W intersects c in a midcube.

Let X be a nonpositively curved cube complex, so that the universal cover $\tilde{X}$ of X is a CAT(0) cube complex. An *immersed hyperplane* $\bar{W}$ of X is the image of a hyperplane W of $\tilde{X}$ under the universal covering projection, and the *immersed carrier* $N(\bar{W})$ of W is the image of $N(W)$.

In [**33**], Sageev proved the following theorem.

Theorem 2.3 (Hyperplane properties). *If W is a hyperplane of the* CAT(0) *cube complex $\tilde{X}$, then*:

1. *W is two-sided, that is, $N(W) \cong W \times [-\frac{1}{2}, \frac{1}{2}]$;*
2. *W is separating, that is, $\tilde{X} - W$ has exactly two components, called halfspaces associated to W;*
3. *any midcube is contained in a unique hyperplane;*
4. *W is a* CAT(0) *cube complex whose hyperplanes are of the form $V \cap W$, where $V \neq W$ is a hyperplane of $\tilde{X}$ that crosses W.*

If $A, B \subset \tilde{X}$ are subspaces of the CAT(0) cube complex $\tilde{X}$ and W is a hyperplane of $\tilde{X}$, then W *separates* A and B if A and B lie in distinct halfspaces associated to W.

Definition 2.4 (Contacting hyperplanes). Let $\tilde{X}$ be a CAT(0) cube complex and V and W a pair of distinct hyperplanes. A 1-cube c is *dual* to W if the 0-cubes of c are separated by W. Equivalently, c is dual to W if W contains the midcube of c.

The hyperplanes V and W *cross* if there is a 2-cube s whose two distinct midcubes are contained in V and W, respectively. This is denoted by $V \perp W$. The hyperplanes V and W *osculate* if they do not cross and there exist distinct 1-cubes c and c', dual to V and W, respectively, such that c and c' have a common 0-cube. In other words, V and W osculate if $N(V) \cap N(W) \neq \emptyset$ and V and W do not cross.

If V and W either cross or osculate, then they *contact*, denoted by $V \perp\!\!\!\!\!{-} W$. Note that $V \perp\!\!\!\!\!{-} W$ if and only if no hyperplane U separates V from W.

The *dimension* of the CAT(0) cube complex $\tilde{X}$ is at least d if $\tilde{X}$ contains a d-cube. If $\tilde{X}$ contains a d-cube but does not contain a $(d+1)$-cube, then $\dim \tilde{X} = d$. Equivalently, $\dim \tilde{X}$ is equal to $\sup_S |S|$, where S varies over all sets of pairwise-crossing hyperplanes. The *degree* of $\tilde{X}$ is at least d if there exists a 0-cube in $\tilde{X}$ with at least d distinct incident 1-cubes. Equivalently, the degree of $\tilde{X}$ is at least d if there is a family of d pairwise-contacting hyperplanes. Hence, the degree of $\tilde{X}$ is bounded below by the dimension.

2.2. *Metric notions*

Let $\tilde{X}$ be a CAT(0) cube complex and let $\mathcal{W}$ be the set of hyperplanes of $\tilde{X}$. Consider the standard path-metric $d_{\tilde{X}}$ on the graph $\tilde{X}^{(1)}$. It is shown in [**12**] that $\tilde{X}^{(1)}$ is a *median graph*: for any three distinct 0-cubes x, y, z, there exists a unique 0-cube $m = m(x, y, z)$ such that

$$d_{\tilde{X}}(x, y) = d_{\tilde{X}}(y, m) + d_{\tilde{X}}(m, x),$$
$$d_{\tilde{X}}(z, y) = d_{\tilde{X}}(y, m) + d_{\tilde{X}}(m, z)$$

and

$$d_{\tilde{X}}(x, z) = d_{\tilde{X}}(z, m) + d_{\tilde{X}}(m, x).$$

From this characterization, or from Theorem 2.3(2), it follows that a path $P \to \tilde{X}^{(1)}$ is a geodesic if and only if P contains at most one 1-cube dual to each hyperplane of $\tilde{X}$. In other words, $d_{\tilde{X}}(x, y)$ counts the number of hyperplanes W such that the 0-cubes x and y lie in distinct halfspaces associated to W.

In this paper, all of our arguments are combinatorial, and we shall work with the metric $d_{\tilde{X}}$ on the median graph $\tilde{X}^{(1)}$. Accordingly, we adopt the following terminology: unless stated otherwise, all paths in $\tilde{X}$ are combinatorial, that is, a path P in $\tilde{X}$ is a continuous combinatorial map $P : I \to \tilde{X}^{(1)}$, where I is a CAT(0) cube complex homeomorphic to an interval. The path P is a *geodesic* if it is a geodesic path in $\tilde{X}^{(1)}$ or, equivalently, if the induced map from the set of hyperplanes of I to the set of hyperplanes of $\tilde{X}$ is injective, that is, if P crosses each hyperplane of $\tilde{X}$ at most once. The subcomplex $Y \subset \tilde{X}$ is *isometrically embedded* (*convex*, *bounded*, etc.) if $Y^{(1)}$ is isometrically embedded (convex, bounded, etc.) in $\tilde{X}^{(1)}$, with respect to $d_{\tilde{X}}$. By, for example, verifying that its 1-skeleton is *gated*, one sees that for each hyperplane H, the carrier $N(H)$ is convex in this sense [**12**].

As mentioned above, there is a piecewise-Euclidean CAT(0) metric on $\tilde{X}$. In Sections 6 and 7, we make several statements about the CAT(0) metric, assuming that $\tilde{X}$ is finite-dimensional. This is justified by the following fact: the space $\tilde{X}$, with its CAT(0) metric, is quasi-isometric to $\tilde{X}^{(1)}$ with the metric $d_{\tilde{X}}$ when $\dim \tilde{X} < \infty$. This fact was proved in greater generality by Bridson [**5**], and a simpler proof in the cubical context appears in [**7**]. The statements about the CAT(0) metric deal with Gromov-hyperbolicity and finite asymptotic dimension, both of which are quasi-isometry invariant properties.

We emphasize, however, that, unless stated otherwise, if we refer to a cubical map $\tilde{Y} \to \tilde{X}$ of CAT(0) cube complexes as an isometric embedding, we mean that the image of $\tilde{Y}$ is a subcomplex whose 1-skeleton is isometrically embedded in $\tilde{X}^{(1)}$ with respect to the combinatorial path-metric $d_{\tilde{X}}$. It is worth noting that Sageev showed that each hyperplane and carrier is convex with respect to the CAT(0) metric [**33**], and these notions of convexity coincide for full subcomplexes [**20**], although we shall not use this fact, or the very natural extension of $d_{\tilde{X}}$ to the whole complex considered by Haglund.

Remark 2.5 (Cubulated groups). If a group G acts on a CAT(0) cube complex $\tilde{X}$ by cubical automorphisms if G acts on $\tilde{X}^{(0)}$, then G stabilizes the set of hyperplanes. In this situation, G acts on the metric space $(\tilde{X}^{(1)}, d_{\tilde{X}})$ by isometries.

The action is *metrically proper* if, for each bounded subcomplex $B \subset \tilde{X}$, there are finitely many $g \in G$ such that $gB \cap B \neq \emptyset$. When $\tilde{X}$ is locally finite, its 1-skeleton is a proper metric space, and thus, a proper action (in the sense that cube stabilizers are finite) is metrically proper. Throughout this paper, a *cubulated group* is one admitting of a metrically proper action by cubical automorphisms on a CAT(0) cube complex.

2.3. *Cubulating wallspaces*

DEFINITION 2.6. A *wallspace* is a pair $(\mathcal{S}, \mathcal{W})$, with $\mathcal{S}$ a (nonempty) set and $\mathcal{W}$ a set of *walls*, which are partitions W of $\mathcal{S}$ into disjoint nonempty *halfspaces* $W^{\pm}$. Moreover, we suppose that for each $s_1, s_2 \in \mathcal{S}$, there is a finite, nonzero number of walls W that *separate* s_1 and s_2, in the sense that s_1 and s_2 lie in distinct halfspaces associated to W.

More generally, W *separates* the subsets $A, B \subset \mathcal{S}$ if A and B lie in distinct halfspaces of W, and W separates the walls U, V if it separates some halfspace of U from some halfspace of V.

Walls $V, W \in \mathcal{W}$ *cross* if each of the four *quarterspaces* $V^{\pm} \cap W^{\pm}$ is nonempty.

REMARK 2.7 (Sageev's construction). A wallspace $(\mathcal{S}, \mathcal{W})$ determines a CAT(0) cube complex $\tilde{X}$ in such a way that the hyperplanes of $\tilde{X}$ correspond to the walls $\mathcal{W}$ and hyperplanes cross if and only if the corresponding walls do.

An *orientation* of W is a choice of exactly one of the halfspaces associated to W, and for each $s \in \mathcal{S}$, to *orient* W *toward* s is to choose the orientation of W that contains s. More generally, for any subset of $\mathcal{S}$ that lies in a single halfspace associated to W, we speak of orienting W toward that subset.

A 0-*cube* is a map $f : \mathcal{W} \to \{W^{\pm} \mid W \in \mathcal{W}\}$ with the following properties.

(1) (Orientation) For each $W \in \mathcal{W}$, we have $f(W) \in \{W^-, W^+\}$, that is, f orients each wall.

(2) (Consistency) For all $V, W \in \mathcal{W}$, we have $f(V) \cap f(W) \neq \emptyset$.

The consistency condition is automatically satisfied for crossing pairs of walls and says that a 0-cube never orients a wall 'away' from another wall. The 0-cube f is *canonical* if there exists $s \in \mathcal{S}$ such that $f(W)$ contains s for each $W \in \mathcal{W}$.

Denote by C_0 the set of all 0-cubes. The 0-cubes $f_1, f_2 \in C_0$ are joined by a 1-cube if and only if there is exactly one wall W such that $f_1(W) \neq f_2(W)$. We thus obtain a graph C_1 whose vertices are the 0-cubes and whose edges are the 1-cubes. In general, C_1 is disconnected, and the cube complex $\tilde{X}$ *dual* to the wallspace $(\mathcal{S}, \mathcal{W})$ is constructed from C_1 as follows.

Choose any canonical 0-cube f_s, which orients each wall toward the element $s \in \mathcal{S}$. If f_t is another canonical 0-cube, then since any two points are separated by finitely many walls, f_s and f_t differ on finitely many walls, and thus belong to the same component of C_1. Denote by $\tilde{X}^{(1)}$ this *canonical component*. One then verifies that $\tilde{X}^{(1)}$ is the 1-skeleton of a uniquely determined CAT(0) cube complex $\tilde{X}$, which is independent of the choice of canonical 0-cube. $\tilde{X}$ is the cube complex *dual* to the wallspace $(\mathcal{S}, \mathcal{W})$ and is completely determined by that data. The set of hyperplanes of $\tilde{X}$ corresponds bijectively to $\mathcal{W}$, and two hyperplanes contact if and only if the corresponding walls are not separated by a third wall. Two hyperplanes cross if and only if the corresponding walls cross.

In general, the noncanonical components of C_1 are 1-skeleta of CAT(0) cube complexes constructed from 'cubes at infinity'; their 0-cubes are consistent orientations of all walls that differ on infinitely many walls from any canonical 0-cube.

The above construction, when $\mathcal{S}$ is a finitely generated group and the walls arise from codimension-1 subgroups, is due to Sageev [**33**]. The general notion of a wallspace was first introduced in [**21**]. Discussions of Sageev's construction in a general wallspace setting appear in [**11**, **24**, **30**].

Sageev's construction is sometimes given in terms of principal ultrafilters on the wallspace $(\mathcal{S}, \mathcal{W})$. We use the notation W^+ and W^- for the halfspaces associated to the wall W. In the following definition, these are merely notation; we are not, in the following definition, designating a map choosing a halfspace for each wall.

DEFINITION 2.8. An *ultrafilter* on the wallspace $(\mathcal{S}, \mathcal{W})$ is a set ω of halfspaces associated to walls in $\mathcal{W}$ subject to the following conditions.

(1) For all walls W, exactly one of the following occurs: $W^+ \in \omega$ or $W^- \in \omega$.

(2) For any pair $W^+ \subset V^+$ of nested halfspaces such that $W^+ \in \omega$, we have $V^+ \in \omega$, and likewise for the other halfspaces associated to V, W.

For $s \in \mathcal{S}$, the *principal ultrafilter* ω_s associated to s is the set of all halfspaces containing s.

Definition 2.8 gives an equivalent construction of the cube complex dual to a wallspace. First, note that any ultrafilter ω on $(\mathcal{S}, \mathcal{W})$ corresponds to a 0-cube of C_1: the inclusion in ω of exactly one halfspace associated to each wall orients all of the walls. The second condition in Definition 2.8 is a paraphrase of the consistency condition on orientations of the set of walls. It is easily seen that the principal ultrafilter ω_s corresponds to the 0-cube that orients each wall toward the element $s \in \mathcal{S}$. Moreover, the 0-cubes corresponding to the ultrafilters ω_1, ω_2 belong to the same component of the graph C_1 if and only if the symmetric difference $\omega_1 \triangle \omega_2$ is finite. Therefore, since any two elements of $\mathcal{S}$ are separated by finitely many walls, any two principal ultrafilters have finite symmetric difference and thus the corresponding 0-cubes belong to the same component of C_1.

We will apply Sageev's construction later to establish a few statements about crossing graphs and contact graphs.

2.4. *Disk diagrams in* CAT(0) *cube complexes*

This subsection summarizes parts of the discussion of disk diagrams in CAT(0) cube complexes appearing in [**36**].

DEFINITION 2.9. Let X be a nonpositively curved cube complex. A *disk diagram* $D \to X$ *in* X is a continuous combinatorial map of cube complexes, where D is a *disk diagram*: a contractible, finite, two-dimensional cube complex equipped with a fixed (topological) embedding into S^2. The *area* of D is the number of 2-cubes in D.

Since D is contractible, the complement of D in S^2 is a 2-cell whose attaching map is the *boundary path* $\partial_p D$ of D. If $D \to X$ is a disk diagram in X, then the restriction of this map to the boundary path of D is a combinatorial path $\partial_p D \to X$. Note that $\partial_p D$ may not be injective on 0-cubes or 1-cubes. If X is simply connected, then any closed combinatorial path in X is the boundary path of a disk diagram $D \to X$.

Fixing an immersed hyperplane W of X, consider the set of midcubes in D that map to W. A maximal concatenation of such midcubes is a *dual curve* C in D mapping to W. Note that each dual curve is a singular curve: each 1-cube in D has at most two incident 2-cubes, and thus each 0-cell of C has valence at most 2, though C may cross itself in the interior of one or more 2-cubes.

A 1-cube of D whose midcube is contained in a dual curve C is *dual to* C. An *end* of a dual curve C is a midpoint of a 1-cube of $\partial_p D$ dual to C. The *carrier* of the dual curve C is the union of closed 2-cubes of D that contain midcubes belonging to C.

A dual curve C with 0 ends is a *nongon*. If C is not a nongon, then it has two ends. A *monogon* is a closed subpath of a dual curve that crosses itself in the initial 2-cube of its carrier, which is equal to the terminal 2-cube. Any dual curve that crosses itself contains a monogon. An *oscugon* is a closed subpath C' of a dual curve C such that C' does not self-cross, such that the

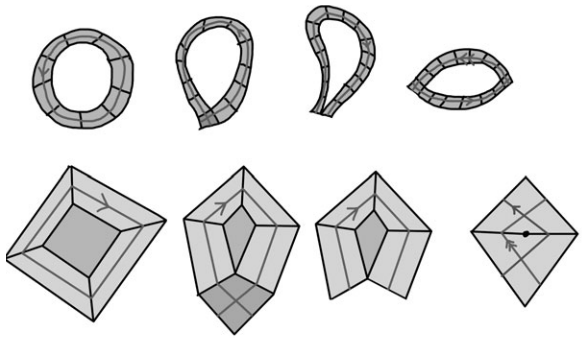

FIGURE 1. *Left to right at the top are heuristic pictures of the carriers of* a: *nongon, monogon, oscugon and bigon. Below each of these figures is an actual disk diagram containing the corresponding configuration. In both sets of pictures, the dual curve itself is decorated with an arrow.*

two distinct terminal 1-cubes of c have a common 0-cube but do not form the corner of a 2-cube in D. A *bigon* is a pair of dual curves that cross in two distinct squares of D; see Figure 1. If C is a dual curve in D whose ends lie on subpaths P, Q of $\partial_p D$, it is often convenient to say that K *emanates* from P and *terminates* on Q (or vice versa), or that K *travels from* P *to* Q.

2.5. *Complexity reductions in disk diagrams*

The techniques used to prove the following lemma are discussed in detail in [**36**] and were developed from ideas of Casson (see [**33**]).

LEMMA 2.10 [**36**]. *Let $P \to X$ be a closed combinatorial path in a nonpositively curved cube complex X and let $D \to X$ be a minimal-area disk diagram among all diagrams D' with $\partial_p D' = P$. Then D contains no nongons, monogons, oscugons or bigons.*

We refer the reader to [**36**] for a discussion of *cancellable pairs* and *hexagon moves* in disk diagrams over nonpositively curved cube complexes, which are used in the proof of Lemma 2.10. In Sections 3 and 4, we will study a particular type of disk diagram and will apply techniques slightly different from those used in the proof of Lemma 2.10. In particular, while a combination of hexagon moves and cancellable pair removals is used in [**36**] to modify a disk diagram without affecting its boundary path, we will sometimes make certain changes to the boundary path, as follows.

Let $H_0 \,\underline{\perp}\, H_1 \,\underline{\perp} \cdots \underline{\perp}\, H_{n-1} \,\underline{\perp}\, H_0$ be (not necessarily pairwise distinct) hyperplanes contacting (at least) as indicated. Then we can choose, for each $i \in \mathbb{Z}_n$, a combinatorial geodesic $P_i \to N(H_i)$ so that there is a closed path $P \to \tilde{X}$ that is the concatenation $P = \prod_{i=1}^{n-1} P_i$. Since $\tilde{X}$ is a CAT(0) cube complex, there is a disk diagram $D \to \tilde{X}$ such that $\partial_p D = P$. This situation is shown schematically in Figure 2.

In our applications, the collection $\{H_i\}$ of hyperplanes is fixed. Given such a collection $\{H_i\}$ of hyperplanes, forming a closed path in the *contact graph* (the intersection graph of the set of hyperplane-carriers), a disk diagram D constructed as above is a *diagram with fixed carriers* for the closed path $\sigma = H_0 \,\underline{\perp}\, H_1 \,\underline{\perp} \cdots \underline{\perp}\, H_{n-1} \,\underline{\perp}\, H_0$. Note that it is possible that $H_i \,\underline{\perp}\, H_j$ for $|i - j| > 1$, but this is not necessarily reflected in D.

The *complexity* $\mathrm{c}(D)$ of D is the pair $(\mathsf{Area}(D), |P|)$, taken in lexicographic order. Suppose that D is of minimal complexity among all diagrams bounded by paths P that decompose in

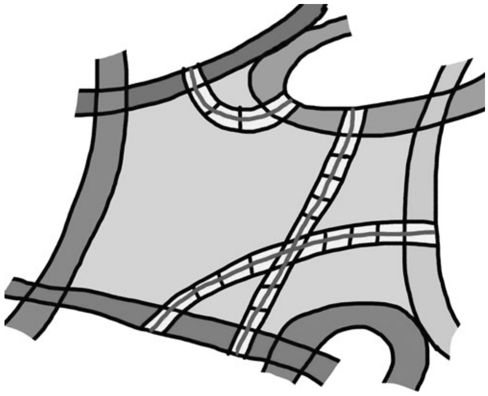

FIGURE 2. *A heuristic picture showing The image of a disk diagram whose boundary path is the concatenation of geodesic segments lying on a fixed collection of hyperplane carriers, along with parts of those carriers. The two configurations of dual curves precluded by Lemma* 2.11 *in the minimal-complexity case are shown.*

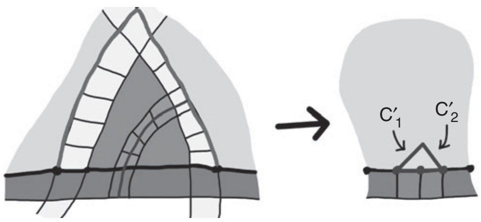

FIGURE 3. *The diagram E arising when two dual curves emanating from P_i cross.*

the above fashion, that is, among all diagrams with fixed carriers for σ. In particular, D is of minimal area among all disk diagrams with boundary path P, so that, by Lemma 2.10, D does not contain any nongons, monogons, oscugons or bigons.

Let K be a dual curve in D. Then K has one end on P_i and the other end on P_j. Since each of P_i and P_j is a geodesic, we must have $i \neq j$, for otherwise P would contain two distinct 1-cubes dual to the same hyperplane, namely the hyperplane to which K maps.

Suppose that K is a dual curve traveling from P_i to P_j, and K' a dual curve traveling from P_i to some P_k, such that K and K' cross in a 2-cube s of D, as at the bottom of Figure 2. Let P'_i be the smallest connected subpath of P_i containing the 1-cubes of P_i dual to K and K'. Let Q and Q' be shortest combinatorial paths in D that start at the ends of P'_i, and travel along the carriers of K and K', respectively, meeting at the corner of s that is separated from P_i by K and K'. Let E be the subdiagram of D bounded by P'_i, Q and Q'; see Figure 3, on the left. If C is a dual curve in E emanating from P'_i, then C cannot end on P'_i by the fact that P'_i is a geodesic, and hence C crosses K or K'. Suppose the former. Then K and C form a triangle of dual curves in D that is properly contained in E. Hence, by choosing an innermost such triangle, we may assume that $|P'_i| = 2$. Hence, P_i contains a path c_1c_2, where c_1 and c_2 are 1-cubes of $N(H_i)$ that form the corner of a 2-cube s in D, as on the right in Figure 3. Let Q_i be the path in $\tilde{X}$ obtained by removing the subpath c_1c_2 from P_i and replacing it by $c'_1c'_2$, where c'_1 is the 1-cube of s opposite to c'_2 and c'_2 the 1-cube of s opposite to c_1.

Note that $|Q_i| = |P_i|$ and Q_i has the same endpoints as P_i; in particular, Q_i is a geodesic. Moreover, Q_i maps to $N(H_i)$. To see this, it suffices to show that c'_1 and c'_2 map to $N(H_i)$. The 1-cubes c_1 and c'_2 are dual to a hyperplane W_1, and c'_1 and c_2 are dual to a hyperplane W_2. If

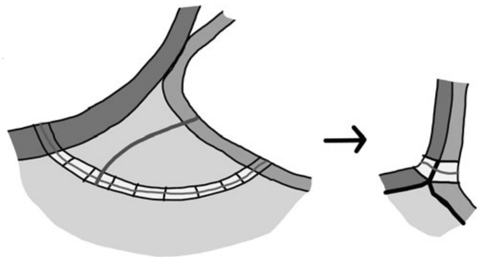

FIGURE 4. *Chopping off a spur using the diagram F.*

$W_1 = H_i$, then c_2' is dual to H_i, whence s, and thus c_1', maps to $N(H_i)$. Hence, suppose that W_1, W_2 and H_i are all distinct. Then W_1, W_2 cross H_i and W_1 crosses W_2 in the 2-cube s. By nonpositive curvature, s lies in a 3-cube of $N(W_1) \cap N(W_2) \cap N(H_i)$ and in particular $Q_i \to N(H_i)$.

By removing s from D and replacing P_i with Q_i, we replace D by a proper subdiagram diagram D' that has fixed carriers for $\{H_i\}$, so that $\mathrm{c}(D') < \mathrm{c}(D)$, contradicting the minimality of D. Hence, no two dual curves in D emanating from any P_i can cross.

Now consider the case in which K emanates from P_i and terminates on P_{i+1}. This is shown at the top of Figure 2 and is enlarged in Figure 4. Let F be the subdiagram of D between and including the carrier of K and the subtended parts of P_i, P_{i+1}. Suppose also that K is innermost, in the sense that no dual curve L in F travels from P_i to P_{i+1} (otherwise, we could argue using L instead of K). If there is a dual curve K' in F, distinct from K, emanating from P_i or P_{i+1}, then K' crosses K in D, contradicting minimality of the complexity, by the preceding argument. Hence, $|K| = 0$ and P_i and P_{i+1} have a common 1-cube c dual to the hyperplane to which K maps, as on the right in Figure 4.

The minimal case is that in which the 1-cube c is a *spur* in the language of [**36**], that is, $\partial_p D$ contains the path cc^{-1}. By removing c from P_i and P_{i+1}, we obtain a subdiagram D' of D that has fixed carriers for σ but that has lower complexity. If not, then since every dual curve in F travels from P_i to P_{i+1}, there is a spur in F, which we find by noting that the terminal 1-cube of P_i coincides with the initial 1-cube of P_{i+1}. In either case, we can remove the part of $P_i P_{i+1}$ between the two occurrences of c and lower the complexity of D while preserving the fixed carriers. We have thus proved the following lemma.

LEMMA 2.11. *Let $\sigma = H_0 \perp\!\!\!\!- H_1 \perp\!\!\!\!- \cdots \perp\!\!\!\!- H_{n-1} \perp\!\!\!\!- H_0$ be a closed path in the contact graph and let D be a diagram with fixed carriers for σ. Suppose that D is of minimal complexity among all diagrams with fixed carriers for σ, and let $\partial_p D = P_0 P_1 \cdots P_{n-1}$ be the boundary path, where each $P_i \to N(H_i)$ is a combinatorial geodesic. Then*:

(1) *for all $i \in \mathbb{Z}_n$, no dual curve K emanating from P_i terminates on $P_{i\pm1}$;*
(2) *if K, K' are dual curves emanating from P_i, then K and K' do not cross.*

We emphasize that it is possible for K to emanate from P_i and terminate on the next positive-length labeled subpath of the path P. More precisely, if $|P_{i+1}| = 0$, then we must still treat P_{i+1} as one of the designated geodesic subpaths of P, since D has fixed carriers. In this case, as above, P_i and P_{i+2} intersect in a spur c mapping to a 1-cube dual to the hyperplane to which K maps, but we cannot remove c, since that would remove the path P_{i+1} (which is an endpoint of c) and result in a diagram that does not have fixed carriers. However, in such a situation, we reach a conclusion that $H_i \perp\!\!\!\!- H_{i+2}$.

2.6. *Crossing graphs and contact graphs*

Unless stated otherwise, graphs in this paper are simplicial in the sense that they have no loops or multi-edges. Also, graphs have the combinatorial metric, with all edges of length 1.

Definition 2.12. If Φ is a subgraph of a graph Λ, then $\mathrm{Full}(\Phi)$ denotes the full subgraph of Λ generated by the vertices of Φ.

Let v be a vertex of Λ and let $n \geqslant 0$. The *full ball* $\bar{B}_n(v) = \mathrm{Full}(\{w \in \Phi^{(0)} : d(v, w) \leqslant n\})$.

The *full sphere* $\bar{S}_n(v)$ denotes the full subgraph of Λ generated by vertices at distance exactly n from v.

We shall often need the following facts about subcomplexes of cube complexes. Let $Y \subset \tilde{X}$ be an isometrically embedded subcomplex of the CAT(0) cube complex $\tilde{X}$. Then for each hyperplane H of $\tilde{X}$, either $H \cap Y = \emptyset$ or $H \cap Y$ is a connected subspace of Y such that $Y - (H \cap Y)$ has two components, one in each halfspace associated to H. In the latter case, *H crosses Y*. Conversely, if $Y \subset \tilde{X}$ is connected and has the property that $H \cap Y$ is connected (or empty) for each hyperplane H, then $Y \hookrightarrow \tilde{X}$ is an isometric embedding.

Indeed, if $Y \cap H$ is connected for each H, then let $a, b \in Y$ be 0-cubes. Let P be a geodesic of Y joining a, b. Suppose that $P = P_1 c_1 P_2 c_2 P_3$, where c_1, c_2 are 1-cubes dual to the same hyperplane H. Then there is a sequence $c_1 = d_0, d_1, d_2, \ldots, d_k, d_{k+1} = c_2$ of 1-cubes, all dual to H, such that d_i and d_{i-1} lie on opposite sides of the same 2-cube of $N(H)$ and the midcube of each d_i lies in $H \cap Y$. Since Y is a subcomplex, each $d_i \subset Y$, and thus there is a geodesic $Q \to N(H) \cap Y$ joining the initial 0-cube of c_1 to the terminal 0-cube of c_2, such that Q does not cross H. On the other hand, $Q c_1 P_2 c_2$ bounds a minimal disk diagram in which all dual curves emanating from Q end on P_2, and thus $|Q| \leqslant |P_2|$. But then $P_1 Q P_3$ is a path in Y joining a, b with $|P_1 Q P_3| < |P|$, a contradiction. Hence, every 1-cube of P is dual to a distinct hyperplane, whence P is a geodesic of $\tilde{X}$ and thus Y is isometrically embedded.

Lemma 2.13 says that a locally convex subcomplex of a CAT(0) cube complex is convex.

Lemma 2.13. *Let $\tilde{X}$ be a* CAT(0) *cube complex with a set $\mathcal{W}$ of hyperplanes. Let $C \subseteq \tilde{X}$ be a connected subcomplex and let $\mathcal{W}'$ be the set of hyperplanes of $\tilde{X}$ that cross C. Then the following are equivalent*:

(1) *the subcomplex $C \subset \tilde{X}$ is convex*;
(2) *distinct hyperplanes $V, W \in \mathcal{W}'$ cross in $\tilde{X}$ if and only if they cross in C, that is, for some 2-cube s dual to V and W, we have $s \subset C$.*

Lemma 2.13 can be proved using minimal-area disk diagrams and hexagon moves; see [**36**].

Remark 2.14. More generally, an isometrically embedded subcomplex C is convex if and only if any pair of hyperplanes V, W of $\tilde{X}$ such that $N(V) \cap C$ and $N(W) \cap C$ are both nonempty contact if and only if $N(V) \cap N(W) \cap C \neq \emptyset$. This is a special case of Helly's theorem for CAT(0) cube complexes, which is stated below and which is discussed in, for example, [**32**]. Helly's theorem appears in many different contexts. For example, convexity and the Helly property are discussed in the context of median spaces in van de Vel's book [**34**].

Chepoi has also pointed out in private communication that Lemma 2.15 also follows from the median property of $\tilde{X}^{(1)}$. Indeed, convex subsets of a median graph are *gated*, and collections of gated subsets enjoy the Helly property.

LEMMA 2.15. *Let $\tilde{X}$ be a* CAT(0) *cube complex and let $Y_1, Y_2, \ldots, Y_n$ be a finite collection of convex subcomplexes of $\tilde{X}$. Suppose that $Y_i \cap Y_j \neq \emptyset$ for all $1 \leqslant i \leqslant j \leqslant n$. Then $\bigcap_i Y_i \neq \emptyset$.*

DEFINITION 2.16 (Contact graph, crossing graph). Let $\tilde{X}$ be a CAT(0) cube complex. The *contact graph* Γ of $\tilde{X}$ is the graph whose vertices are the hyperplanes of $\tilde{X}$, with hyperplanes V and W joined by an edge if and only if $V \perp\!\!\!\!- W$. Equivalently, Γ is the nerve of the covering of $\tilde{X}$ by the set of hyperplane carriers. The *crossing graph* Δ of $\tilde{X}$ is the subgraph of Γ containing all of the vertices, with V and W joined by an edge exactly when $V \perp W$. When discussing these graphs, the terms 'vertex' and 'hyperplane' are used interchangeably.

While Γ is always connected, Δ may not be, as in the following example.

EXAMPLE 2.17. When $\tilde{X}$ is a tree, the hyperplanes are the midcubes of the edges. The vertices of Γ correspond to the 1-cubes of $\tilde{X}$, with two vertices adjacent exactly when the corresponding 1-cubes have a common 0-cube. In particular, for each 0-cube of $\tilde{X}$, the contact graph contains a complete graph whose vertex-set has cardinality equal to the valence of that 0-cube. The crossing graph Δ has no edges.

If $\tilde{X}$ is a cube, then $\Delta = \Gamma$ is the complete graph on the set of midcubes.

If $\tilde{X}$ is the standard tiling of $\mathbb{R}^n$ by n-cubes, then Δ is a complete n-partite graph with each class of the n-partition an order-isomorphic copy of $\mathbb{Z}$, where the hyperplanes in each class are ordered by designating a halfspace for each in such a way that the designated halfspaces are totally ordered by inclusion. Adding to Δ an edge between consecutive vertices in each class gives the contact graph Γ. More generally, if $\tilde{X}$ and $\tilde{Y}$ are CAT(0) cube complexes, then the contact graph of $\tilde{X} \times \tilde{Y}$ is the join of the contact graphs of the factors.

The following basic notion is useful in Section 7.

DEFINITION 2.18 (Inseparable set of hyperplanes). Let $\mathcal{W}$ be the set of hyperplanes in the CAT(0) cube complex $\tilde{X}$. The set $\mathcal{W}' \subseteq \mathcal{W}$ is *inseparable* if, for any two $W_1, W_2 \in \mathcal{W}'$, no hyperplane $W_3 \in \mathcal{W} - \mathcal{W}'$ separates W_1 from W_2.

The following proposition shows that the class of graphs that are crossing graphs of CAT(0) cube complexes is very large.

PROPOSITION 2.19. *For any simplicial graph Δ, there exists a* CAT(0) *cube complex $\tilde{X}$ whose crossing graph is Δ.*

Proof. First suppose that Δ is connected and does not consist of a single vertex. We first construct a wallspace from Δ.

For each $v \in \Delta^{(0)}$, let $I(v)$ be a set of vertices with the same cardinality as the set of vertices of Δ adjacent to v, together with two additional vertices $a(v), b(v)$. There is an *augmented graph* $\Delta^\sharp$ formed by inflating each vertex of Δ into a disjoint set of vertices according to the valence of v. More precisely, $\Delta^\sharp$ is the graph whose vertices are $\coprod_{v \in \Delta^{(0)}} I(v)$, and whose edges are as follows. If v and w are adjacent vertices of Δ, then join some vertex of $I(v)$ to some vertex of $I(w)$ by an edge, and do this in such a way that the resulting graph has the property that all vertices in each $I(v) - \{a(v), b(v)\}$ have exactly one incident edge; the remaining vertices have valence 0. Write $e \sim f$ when e and f are adjacent in $\Delta^\sharp$.

The underlying set of the wallspace is $S = (\Delta^\sharp)^{(0)}$. For each $w \in \Delta^{(0)}$, define a wall (w^+, w^-) by

$$w^+ = (I(w) - \{b(w)\}) \cup \{f : \exists e \in I(w), e \sim f\},$$
$$w^- = S - w^+.$$

By construction, two walls in $\mathcal{W}$ cross if and only if the corresponding vertices of Δ are adjacent. Indeed, let v and w be adjacent vertices of Δ. Then $w^+ \cap v^+$ contains the vertices of $I(v) \cup I(w)$ corresponding to the endpoints of the edge of Δ joining v and w. The intersection of v^+ and w^- consists of the elements of $I(w)$ that do not correspond to the edge joining v to w. The extra element $a(w)$ of $I(w)$ guarantees that there is at least one of these. On the other hand, if v and w are nonadjacent, then $(I(w) - b(w)) \cap \{f : \exists e \in I(v),\, e \sim f\} = \emptyset$, so $v^+ \cap w^+ = \emptyset$, since $I(v) \cap I(w) = \emptyset$ for all v, w. Finally, w^- contains $b(w)$, by definition. On the other hand, $b(w) \notin I(v)$, since $v \neq w$, and $b(w)$ is not adjacent to any vertex in $I(v)$, so that $b(w) \in v^-$. Hence, $w^- \cap v^- \neq \emptyset$.

The cube complex $\tilde{X}$ dual to $(S, \mathcal{W})$, therefore, has crossing graph Δ. Indeed, two hyperplanes of $\tilde{X}$ cross if and only if the corresponding walls cross. If Δ consists of a single vertex, then define $\tilde{X}$ to be a single 1-cube.

If Δ_1 and Δ_2 are distinct components of Δ, then the preceding construction can be performed independently on each component that has more than one vertex, and the resulting cube complexes attached along a single 0-cube, adding osculations, but not crossings, of hyperplanes. In fact, every CAT(0) cube complex with disconnected crossing graph consists of a collection of cube complexes with connected crossing graphs, glued along various 0-cubes. □

The proof of Proposition 2.19 shows that Δ does not uniquely determine $\tilde{X}$ if Δ is disconnected, but this nonuniqueness can happen in other ways. For example, consider $\Delta = K_{2,3}$, a complete bipartite graph with five vertices. Then Δ is the crossing graph of $[-1, 1] \times [-2, 1]$, and is also the crossing graph of $T \times [-1, 1]$, where T is a tripod. However, these two complexes have different contact graphs; one is the join of two line segments and one is the join of a line segment and a triangle.

PROPOSITION 2.20 (Recubulation). *Let $\tilde{X}$ be a* CAT(0) *cube complex with contact graph* Γ. *There exists a* CAT(0) *cube complex $\tilde{X}_r$ whose crossing graph is equal to* Γ, *and there is an isometric embedding $\tilde{X} \to \tilde{X}_r$.*

Proof. The larger cube complex $\tilde{X}_r$ is constructed from $\tilde{X}$ by *recubulating*. For each edge $V \perp\!\!\!\!\!\smile W$ of Γ, there is a set $\{(c, c')\}$ of all pairs of 1-cubes such that c is dual to V and c' is dual to W, such that c and c' meet in a 0-cube. Since hyperplanes in a CAT(0) cube-complex do not self-osculate, each pair (c, c') determines a unique 0-cube $c \cap c'$. For each such pair (c, c') corresponding to an osculation, attach a square s to $\tilde{X}$ by gluing two consecutive edges of s along cc' and let $\tilde{X}'$ be the (possibly not nonpositively curved) auxiliary cube complex obtained from $\tilde{X}$ by attaching all such squares for all osculation-edges $V \perp\!\!\!\!\!\smile W$ in Γ. Each hyperplane W of $\tilde{X}$ extends to a subspace $W' \subset \tilde{X}'$ that separates $(\tilde{X}')^{(0)}$ into two disjoint subsets. Indeed, the midcube of s dual to c is added to V and likewise for c' and W.

The correspondence $W \mapsto W'$ is bijective, since the 1-cubes of each new square s are in two distinct parallelism classes corresponding to the original pair of osculating hyperplanes, which were distinct. Hence, $((\tilde{X}')^{(0)}, \{W'\} \cong \Gamma^{(0)})$ is a wallspace with the property that two walls W' and V' cross if and only if the corresponding vertices in Γ are adjacent. Cubulating this wallspace gives the desired CAT(0) cube complex $\tilde{X}_r$; see Figure 5. Note that $\tilde{X}$ isometrically embeds in $\tilde{X}_r$. □

REMARK 2.21. Proposition 2.20 can be proved more topologically by noting that $\tilde{X}'$ deformation retracts to $\tilde{X}$ and is thus simply connected. Higher-dimensional cubes can be added to $\tilde{X}'$ where necessary to produce the CAT(0) cube complex $\tilde{X}_r$ whose hyperplanes correspond to those of $\tilde{X}$, since the hyperplanes of a CAT(0) cube complex are determined

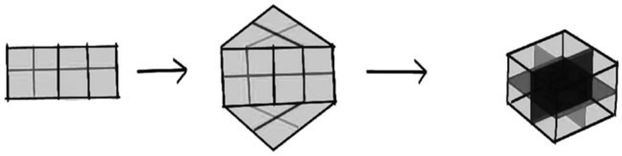

FIGURE 5. *Turning osculations into crossings. The two vertical hyperplanes osculate on the left. For each of the two pairs of intersecting* 1*-cubes corresponding to this osculation, we add a* 2*-cube, to obtain the middle picture. To maintain nonpositive curvature, we must now add a* 3*-cube, and we obtain the picture on the right.*

by the 1-skeleton and each parallelism-class of 1-cubes (explained in, for example, [**12**, **22**]) is already represented in $\tilde{X}$ and thus in $\tilde{X}'$. Each osculation of hyperplanes in $\tilde{X}$ is replaced by a crossing in $\tilde{X}_r$ and the dimension of $\tilde{X}_r$ is thus equal to the cardinality of the largest clique in Γ.

3. *Full spheres in contact graphs*

Recall that the *full sphere* $\bar{S}_n(V) \subseteq \Gamma$ is the full subgraph of Γ generated by hyperplanes at distance exactly n from V in Γ.

DEFINITION 3.1 (Roots of a full sphere). Let $\bar{S}_n(V)$ be a full sphere in Γ, with $n \geqslant 1$. A *grade-n root* C of $\bar{S}_n(V)$ is the full subgraph of Γ generated by hyperplanes in $\bar{S}_n(V)^{(0)} \cap B$, where B is a path-component of $\Gamma - \bar{B}_{n-1}(V)$. The grade-0 root is the vertex corresponding to V.

A root C of $\bar{S}_n(V)$ is a union of path-components of $\bar{S}_n(V)$. The 0-skeleta of the roots of $\bar{S}_n(V)$ may be regarded as equivalence classes, where hyperplanes V and W are equivalent if they are joined by a path in Γ that contains no vertex of $\bar{B}_{n-1}(V)$. The language of *graded* hyperplanes defined below facilitates discussion of full spheres.

DEFINITION 3.2. Let Γ be the contact graph of the CAT(0) cube complex $\tilde{X}$. With respect to a fixed base hyperplane V^0, the hyperplane W has *grade* n if $W \in \bar{S}_n(V^0)$. If $D \to \tilde{X}$ is a disk diagram containing a dual curve K, the *grade* of K is the grade of the hyperplane to which K maps.

3.1. *Precursors, ancestors and footprints*

Precursors are local features of Γ that govern how concentric full spheres fit together, and footprints are related subspaces of $\tilde{X}$ by which the presence of grade-n hyperplanes are reflected in the grade-$(n-1)$ hyperplanes. Ancestors are subcomplexes of $\tilde{X}$ that contain precursors and footprints. Precursors have an implicit role in the proof that Γ is a quasi-tree.

DEFINITION 3.3 (Planar grid). Let $\mathbb{R}$ denote the real line, regarded as a cube complex with $\mathbb{R}^{(0)} = \mathbb{Z}$. An *interval* I is a nonempty connected subcomplex of $\mathbb{R}$. A *planar grid* S is a two-dimensional CAT(0) cube complex isomorphic to $I \times I'$, where I, I' are (possibly infinite) subdivided intervals. Note that a planar grid is a convex subcomplex of $\mathbb{R} \times \mathbb{R}$. Planar grids feature in a minor manner in Lemma 3.5 and play an important role in Section 7.

DEFINITION 3.4. Fix a base hyperplane V of $\tilde{X}$ and grade the hyperplanes of $\tilde{X}$ with respect to V. Let $U \in \bar{S}_n(V)$, with $n \geqslant 1$. A *precursor of U* is a hyperplane $W \in \bar{S}_{n-1}(V)$

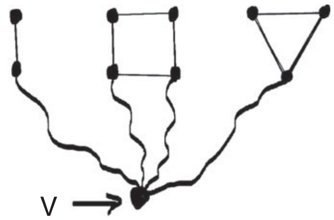

Figure 6. *Left to right, in* Γ: *a precursor, an edge-precursor and a common precursor. The wavy paths are geodesics of length* $n-1$.

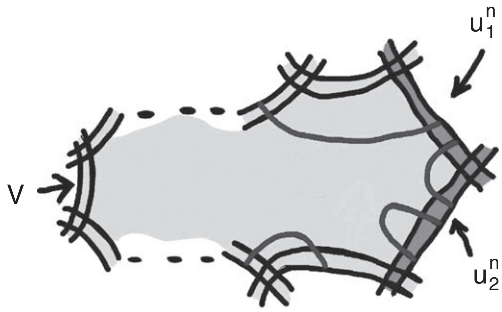

Figure 7. *The disk diagram in Lemma* 3.5.

such that $U \perp\!\!\!\!\!\!\!- W$. For $n \geqslant 1$, a *common precursor* for an edge $U_1 \perp\!\!\!\!\!\!\!- U_2$ in $\bar{S}_n(V)$ is a vertex $W \in \bar{S}_{n-1}(V)$ such that any length-n path from V to U_i passes through W, for $i = 1, 2$. For example, all edges of $S_1(V)$ have V as a common precursor of their endpoints.

For $n \geqslant 2$, an *edge-precursor* for an edge $U_1 \perp\!\!\!\!\!\!\!- U_2$ in $\bar{S}_n(V)$ is an edge $W_1 \perp\!\!\!\!\!\!\!- W_2$ in $\bar{S}_{n-1}(V)$ such that $U_i \perp\!\!\!\!\!\!\!- W_i$ for $i = 1, 2$; see Figure 6.

The following lemma shows that edge-precursors and common precursors exist in Γ. The edge in $S_{n-1}(V)$ defining an edge-precursor may arise as an osculation: the analogous statement for crossing graphs is false.

Lemma 3.5. *Let* $\tilde{X}$ *be a* CAT(0) *cube complex with contact graph* Γ. *For* $n \geqslant 2$, *if* U_1^n *and* U_2^n *in* $\bar{S}_n(V) \subset \Gamma$ *are adjacent, then either they have a common precursor or the edge* $U_1^n \perp\!\!\!\!\!\!\!- U_2^n$ *has an edge-precursor.*

Proof. Either the U_i^n have a common precursor or there exist geodesic paths σ_i in Γ, for $i = 1, 2$, which are concatenations $V = U_i^0 \perp\!\!\!\!\!\!\!- U_i^1 \perp\!\!\!\!\!\!\!- \cdots \perp\!\!\!\!\!\!\!- U_i^{n-1} \perp\!\!\!\!\!\!\!- U_i^n$ such that $U_i^j \in S_j(V)$ and $U_1^{n-1} \neq U_2^{n-1}$. In the latter case, choose a closed path $\gamma \to \tilde{X}$ that is a concatenation

$$\gamma = P^0 P_1^1 P_1^2 \cdots P_1^{n-1} P_1^n P_2^n P_2^{n-1} \cdots P_2^1,$$

where $P_i^j \to N(U_i^j)$ and $P^0 \to N(V)$. Let $D \to \tilde{X}$ be a disk diagram with boundary path γ, and suppose that D has minimal complexity among all such diagrams for all such choices of geodesic segments in Γ. This situation is illustrated in Figure 7. No dual curve in D has both ends on a subpath of γ that maps to a single hyperplane carrier, by minimality of area.

Let C be a dual curve originating on P_i^n. Since U_i^n has grade n, the hyperplane U to which C maps cannot cross U_k^j for $j < n-2$, so that C must end on P_k^j with $k = 1, 2$ and $j \geqslant n-2$.

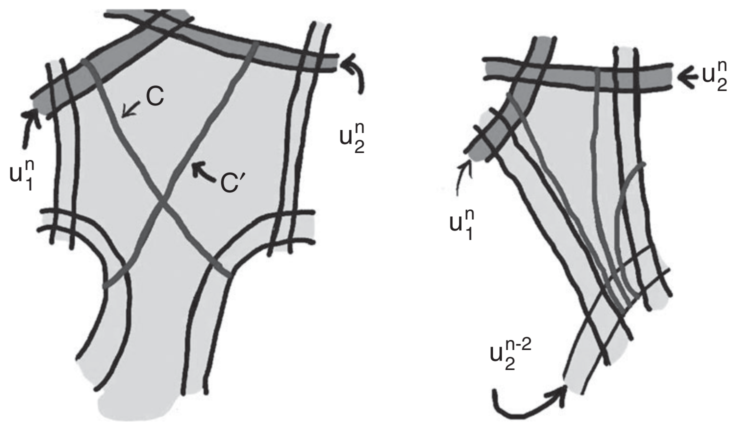

FIGURE 8. *Obtaining a precursor-pair. The dual curves on the right are* C_1, C_2, C_3.

If C ends on P_i^{n-1}, then there is a lower-complexity choice of D by Lemma 2.11. If C ends on P_i^{n-2}, then the path σ_i can be modified by replacing U_i^{n-1} by U, leading to a lower-area disk diagram. Hence, C ends on P_k^j with $j = n-1$ or $n-2$ and $k \neq i$.

If C ends on P_k^{n-2}, as on the left of Figure 8, then there are two possibilities. If some dual curve C' originating on P_k^n ends on P_i^{n-2}, then the hyperplanes corresponding to C and C' are an edge-precursor for U_1^n and U_2^n. If not, then observe that σ_i can be replaced by the path $U_k^0 \perp\!\!\!\!\!- U_k^1 \perp\!\!\!\!\!- \cdots \perp\!\!\!\!\!- U \perp\!\!\!\!\!- U_i^n$, yielding a lower-area *pentagonal diagram* D' as on the right of Figure 8. Any dual curve to the subpath of P_k^{n-2} contained in $\partial_p D'$ leads to a contradiction: if such a dual curve C_1 ends on P_i^n, then area can be further decreased by using C_1 in place of C; if C_2 travels from P_k^{n-2} to P_k^n, then replace U_k^{n-1} by the hyperplane corresponding to C_2; if C_3 has any of the other two possible destinations, then Lemma 2.11 gives a contradiction. These possibilities are shown on the right in Figure 8. Hence, the subtended part of P_k^{n-2} is a trivial path, and $N(U) \cap N(U_k^{n-1}) \neq \emptyset$, so that those hyperplanes form an edge-precursor. The remaining possibility is that all dual curves emanating from P_i^n end on P_k^{n-1} and vice versa. No two dual curves from P_i^n or P_i^{n-1} cross and, thus, there is a planar grid in D, as in Figure 9. An innermost dual curve C to P_i^{n-1} that does not end on P_k^n forms part of the boundary path of a subdiagram $D' \subset D$, containing the planar grid, such that any dual curves in D' emanating from C have no possible destination. Thus, $U_1^{n-1} \perp\!\!\!\!\!- U_2^{n-1}$. □

REMARK 3.6. The analog of Lemma 3.5 does not hold for crossing graphs. Consider a 10-gon tiled by squares consisting of five squares meeting around a 0-cube. Any choice of base hyperplane gives an adjacent pair of grade-2 (in the crossing graph) hyperplanes that do not have a common (crossing) precursor of grade 1 or an edge-precursor, since the grade-1 hyperplanes do not cross.

Given a central hyperplane V and a radius $n \geqslant 0$, there is a subcomplex $Y_n = \bigcup_{W^n} N(W^n)$ corresponding to $\bar{S}_n(V)$. For $n \geqslant 1$, the subcomplex $Y_n \subset \tilde{X}$ is not in general convex, but nonetheless exhibits some of the behavior of a convex subcomplex.

DEFINITION 3.7 (Ancestor). Given $U \in \bar{S}_n(V)^{(0)}$, the *ancestor* of U, Ancestor(U), is the subcomplex of Y_{n-1} consisting of the union of all carriers $N(W)$ such that $W \in \bar{S}_{n-1}(V)^{(0)}$ and $U \perp\!\!\!\!\!- V$.

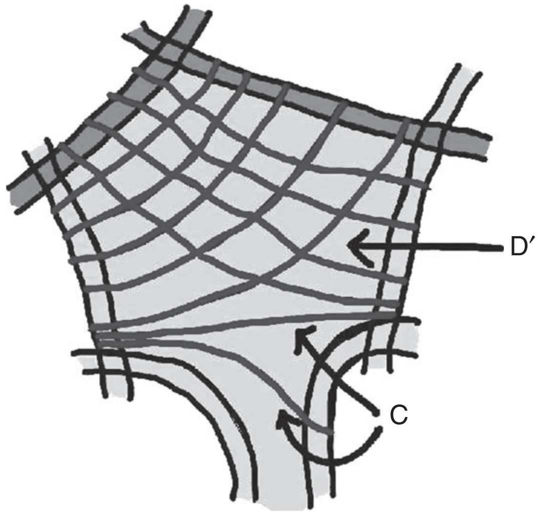

FIGURE 9. *The grid case.*

DEFINITION 3.8 (Footprint). For $n \geqslant 1$, if $U \in \bar{S}_n(V)^{(0)} \subset \Gamma$, then the *footprint* $F(U)$ *of* U *in* $\bar{S}_{n-1}(V)$ is the subspace

$$F(U) = \bigcup_{W \in S_{n-1}(V)^{(0)}} N(U) \cap N(W)$$

of Ancestor(U). Each intersection $N(U) \cap N(W) = F(U;W)$ is the *footprint of* U *in* W.

The following lemmas enable statements about hyperplanes to be proved by induction on dimension, since they show that hyperplanes inherit the adjacency properties of their footprints.

LEMMA 3.9. *For* $U \in \bar{S}_n(V)^{(0)}$, *the ancestor,* Ancestor(U), *and the footprint,* $F(U)$, *are connected.*

Proof. If $n = 1$, then the ancestor is the connected subcomplex $N(V)$ and the footprint $N(U) \cap N(V)$ is connected by convexity of hyperplane carriers.

Let U_1^{n-1} and U_2^{n-1} be distinct precursors of U. For $i = 1, 2$, choose geodesics

$$V = U_i^0 \perp\!\!\!\!\!\!- \cdots \perp\!\!\!\!\!\!- U_i^{n-1} \perp\!\!\!\!\!\!- U,$$

in Γ. As in Lemma 3.5, choose a closed path

$$\gamma = P_0 P_1^1 P_1^2 \cdots P_1^{n-1} Q P_2^{n-1} \cdots P_2^1,$$

with $P_0 \to N(V)$, $P_i^j \to N(U_i^j)$ and $Q \to N(U)$. Let $D \to \tilde{X}$ be a disk diagram with boundary path γ, and suppose that the choices of precursors, geodesics in Γ, γ and of D are made in such a way that D has minimal complexity with respect to all these possibilities; see Figure 10. Consider a dual curve C in D with an end on Q. Every possibility for the other end of C leads to a contradiction: two ends on Q gives a bigon; an end on P_i^{n-1} leads to a contradiction of Lemma 2.11 (the osculating case is shown at the right of Figure 10); an end on P_i^{n-2} leads to a modification of geodesic in Γ resulting in an area reduction; an end on P_i^{n-k} with $k > 2$ contradicts the hypothesis that $U \in \bar{S}_n(V)$. An end on P_0 leads to a closer pair of precursors

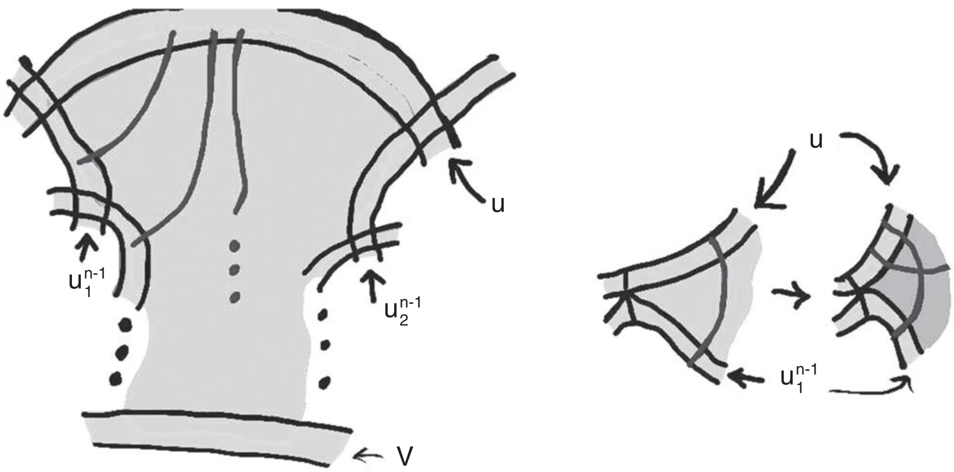

FIGURE 10. *Ancestors are connected.*

and a choice of geodesic in Γ that lowers area for $n = 2$, and contradicts the fact that the chosen path in Γ is geodesic if $n > 2$. Hence, Q is a length-0 path, so that $N(U_1^{n-1}) \cap N(U_2^{n-1}) \neq \emptyset$. The preceding argument also proves connectedness of $F(U)$. $\square$

LEMMA 3.10. *If $U_1, U_2 \in \bar{S}_n(V)^{(0)}$ and $W \in S_{n-1}(V)$ is a common precursor, then $U_1 \perp U_2$ if and only if $F(U_1; W) \cap F(U_2; W) \neq \emptyset$.*

Proof. This follows immediately from Lemma 2.15. $\square$

4. *Contact graphs are quasi-trees*

Fix a base hyperplane V^0 of $\tilde{X}$. For each $n \geqslant 0$, let $\mathcal{C}^n$ denote the set of grade-n roots of the full sphere $\bar{S}_n(V^0)$. Recall that a root $C \in \mathcal{C}^n$ is the full subgraph of Γ generated by the vertices $V^n \in \bar{S}_n(V^0)$ with the property that any two $V_1^n, V_2^n \in C$ are joined by a path in $\Gamma - \bar{B}_{n-1}(V^0)$. In particular, the graph C may not be connected.

The main theorem in this section is Theorem 4.1, and we give two proofs that are quite different.

THEOREM 4.1. *Let $\tilde{X}$ be a* CAT(0) *cube complex with contact graph Γ. Then Γ is quasi-isometric to a tree.*

Proof of Theorem 4.1 *using the bottleneck criterion.* Manning's 'bottleneck' criterion, introduced in [**27**], is as follows:

The geodesic metric space (Y, d) is quasi-isometric to a simplicial tree if and only if there exists $\delta > 0$ such that, for any two points $x, y \in Y$, there exists a midpoint $M = M(x, y)$ such that $d(M, x) = d(M, y) = \frac{1}{2}d(x, y)$ and any path joining x to y contains a point within δ of M.

Let V_0, V_n be hyperplanes, and let $\{V_i\}_{i=1}^{n-1}$ be the set of hyperplanes V_i such that V_0 and V_n lie in distinct halfspaces associated to V_i, that is, the *set of hyperplanes separating* V_0 *and* V_n. Then, for each i, any path in Γ joining V_0 to V_n must contain either V_i or some hyperplane that crosses V_i. Indeed, if $V_0 = W_0 \perp W_1 \perp \cdots \perp W_m = V_n$ is a path, then there exists a path $Q = Q_0Q_1 \cdots Q_n$ in $\tilde{X}$, where each Q_j lies in $N(W_j)$. Now, since V_i separates W_0 from W_m, the path Q must contain a 1-cube c dual to V_i. For some j, the path Q_j contains c. Either $W_j = V_i$ and c is dual to W_j, or $c \subset N(W_j)$ is not dual to W_j, and hence V_i and W_j cross.

Let $V_0 = U_0 \perp U_1 \perp \cdots \perp U_m = V_n$ be a geodesic path in Γ joining V_0 to V_n. Let M be the midpoint of this path, so that either $M = U_{m/2}$ or M is the midpoint of $U_{(m-1)/2} \perp U_{(m+1)/2}$, according to the parity of m. There exists $i \leqslant n-1$ such that $d_\Gamma(M, V_i) \leqslant \frac{3}{2}$. Indeed, either $U_{m/2} = M$ is equal to, or crosses, some V_i, or $U_{(m\pm 1)/2}$ is equal to, or crosses, some V_i, by Lemma 4.2. Here, we have assumed that $m \geqslant 2$, in order to apply Lemma 4.2. If $m = 1$, then $U_0 \perp U_m$, and the midpoint M of this edge obviously satisfies Manning's criterion with $\delta = \frac{1}{2}$.

Let $V_0 = W_0 \perp W_1 \perp \cdots \perp W_p = V_n$ be some other path joining V_0 to V_n. Then some W_j either crosses or coincides with V_i, so that $d_\Gamma(W_j, M) \leqslant d_\Gamma(W_j, V_i) + d_\Gamma(V_i, M) \leqslant \frac{5}{2}$. Thus, Manning's criterion is verified with $M(V_0, V_n) = M$ and $\delta = \frac{5}{2}$. □

LEMMA 4.2. *Let $U_0 \perp U_1 \perp \cdots \perp U_m$ be a geodesic of Γ and let $\{V_i\}_{i=0}^{n-1}$ be the set of hyperplanes separating U_0 from U_m. Then for each j with $1 \leqslant j \leqslant m-1$, there exists i such that $d_\Gamma(U_j, V_i) \leqslant 1$.*

Proof. Let Q be a geodesic segment beginning on $N(U_0)$ and ending on $N(U_m)$, chosen as short as possible, so that the set of hyperplanes that are dual to 1-cubes of Q is exactly $\{V_i\}$. For $0 \leqslant k \leqslant m$, let $P_k \to N(U_k)$ be a geodesic segment, chosen so that the P_k are concatenable, that is, there is a path $P = P_0P_1 \cdots P_m \to \tilde{X}$, and suppose that P joins the endpoints of Q. Let $D \to \tilde{X}$ be a disk diagram bounded by P and Q, and suppose that the choices of P, Q, D are made so that $\mathsf{Area}(D)$ is as small as possible. Choose $j \leqslant m$, and consider the dual curves in D that emanate from P_j. There must be at least one such dual curve, since otherwise $U_{j-1} \perp U_{j+1}$, contradicting the fact that $U_0 \perp \cdots \perp U_m$ is a geodesic. If K is a dual curve that emanates from P_j and ends on Q, then K maps to a hyperplane V_i that separates U_0 from U_p, whence $d_\Gamma(U_j, V_i) = 1$. In this case, the proof is complete: we have found a separating hyperplane V_i that crosses U_j.

Otherwise, each such K ends on P_s for some $s \leqslant m$. Now, $s \neq j$, since P_j is a geodesic, and $|s - j| \neq 1$, since otherwise, if K traveled from P_j to P_{j+1}, then we could perform hexagon moves and removal of spurs to choose a smaller choice of D, fixing the carriers $\{N(U_k)\}$. On the other hand, if K were to travel from P_j to P_s with $|s-j| \geqslant 3$, then there would be a path $U_s \perp W \perp U_j$ in Γ, where W is the hyperplane to which K maps, showing that $d_\Gamma(U_j, , U_s) \leqslant 2$, a contradiction. Hence, assume that every dual curve emanating from P_j ends on P_{j+2} or P_{j-2}. Moreover, by minimality of area, no two such dual curves cross. Now, label the dual curves $K_1, \ldots, K_c$ so that K_q is dual to the qth 1-cube of P_j, measuring from $P_j \cap P_{j-1}$ to $P_j \cap P_{j+1}$. If each K_q ends on P_{j-2}, then $U_{j-2} \perp W_c \perp U_{j+1}$, where W_c is the hyperplane to which K_c maps, and this is a contradiction. Similarly, if each K_q ends on P_{j+2}, then $U_{j-1} \perp W_1 \perp U_{j+2}$, another contradiction. Hence, we have $1 < a < c$ such that $K_1, \ldots, K_a$ travel from P_j to P_{j-2} and $K_{a+1}, \ldots, K_c$ travel from P_j to P_{j+2}. But then $U_{j-2} \perp W_a \perp W_{a+1} \perp U_{j+2}$, contradicting the fact that $d_\Gamma(U_{j-2}, U_{j+2}) = 4$. Hence, some K_q must end on Q and map to a hyperplane separating U_0 from U_m; see Figure 11. □

Theorem 4.1 is also provable using disk diagrams.

Proof of Theorem 4.1 *using hyperplane grading.* Fix a base vertex V^0 of Γ. The resulting *graded root-tree* $\mathcal{T}$ is the following graph. The 0-skeleton of $\mathcal{T}$ is the set $\coprod_{n \geqslant 0} \mathcal{C}^n$. Edges join

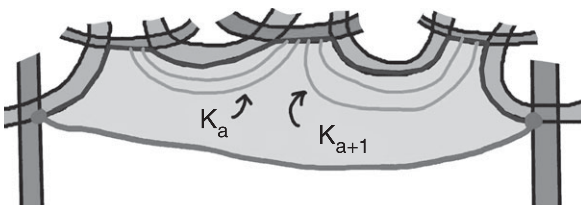

FIGURE 11. *The dual curves emanating from P_j cannot all end on P.*

vertices in $\mathcal{C}^n$ to vertices in $\mathcal{C}^{n+1}$. Precisely, if $C^n \in \mathcal{C}^n$ and $C^{n+1} \in \mathcal{C}^{n+1}$, then C^n is adjacent to C^{n+1} if and only if C^n contains a vertex of Γ that is adjacent to a vertex of $C^{n+1} \subset \Gamma$.

The graph $\mathcal{T}$ is a tree. To see this, note that for each $n \in \mathbb{N}$, no two vertices in $\mathcal{C}^n$ are adjacent, so that the presence of a cycle in $\mathcal{T}$ implies that for some n, there is a $C^{n+1} \in \mathcal{C}^{n+1}$ that is adjacent to two distinct vertices $C_1^n, C_2^n \in \mathcal{C}^n$. It follows that there are hyperplanes $V_i^{n+1} \in C^{n+1}$, $V_i^n \in C_i^n$ for $i = 1, 2$ such that $V_i^n \perp\!\!\!\!\perp V_i^{n+1}$. By definition, V_1^{n+1} and V_2^{n+1} are joined by a path in $\Gamma - \bar{B}_n(V^0)$, and we thus have a path in $\Gamma - \bar{B}_{n-1}(V^0)$ joining the V_i^n, so that $C_1^n = C_2^n$, a contradiction.

The graph Γ is quasi-isometric to $\mathcal{T}$. Indeed, consider the map $\phi : \Gamma \to \mathcal{T}$ such that ϕ sends each hyperplane V^n to the unique root of $\bar{S}_n(V^0)$ containing it and does likewise for edges that have both endpoints in the same full sphere. The remaining edges of Γ join hyperplanes in roots of $\bar{S}_n(V^0)$ to $\mathcal{T}$-adjacent roots of $S_{n+1}(V^0)$, for $n \geqslant 0$. These edges map isometrically to the corresponding edges of $\mathcal{T}$. The map ϕ is surjective and a quasi-isometric embedding by Lemma 4.3. $\square$

Lemma 4.3 asserts the existence of a uniform bound on the diameters of the roots in the graded graph Γ.

LEMMA 4.3. *There exists a constant M such that for any $n \geqslant 0$ and any base hyperplane V^0, if $C \in \mathcal{C}^n$, then $\operatorname{diam}_\Gamma(C) \leqslant M$.*

Proof. Argue by induction on the grade n of $C = C^n$. Since $\mathcal{T}$ is a tree, there is a unique sequence $C^0, C^1, \ldots, C^n$ of roots joining C^0 to C^n, that is, for $0 \leqslant i \leqslant n-1$, if $V^{i+1} \in C^i$ and V^i is a precursor of C^{i+1}, then $V^i \in C^i$.

Let $V_1^n, V_2^n \in C^n$. By definition, there is a path

$$\rho = V_1^n \perp\!\!\!\!\perp U_1 \perp\!\!\!\!\perp U_2 \cdots \perp\!\!\!\!\perp U_m \perp\!\!\!\!\perp V_2^n,$$

in Γ of minimal length so that U_i has grade at least n for $1 \leqslant i \leqslant m$. For $i \in \{1, 2\}$, choose Γ-geodesics

$$\sigma_i = V^0 \perp\!\!\!\!\perp V_i^1 \perp\!\!\!\!\perp \cdots \perp\!\!\!\!\perp V_i^{n-1} \perp\!\!\!\!\perp V_i^n,$$

joining V^0 to V_i^n. Note that for each $k \leqslant n$, the hyperplane V_i^k has grade exactly k and lies in C^k. For each $i \in \{1, 2\}$ and each $k \leqslant n$, choose a geodesic segment $P_i^k \to N(V_i^k)$, and for $1 \leqslant j \leqslant m$, choose a geodesic segment $Q_j \to N(U_j)$ in such a way that the above geodesics are concatenable, that is, there is a closed path

$$P = P_0 P_1^1 P_1^2 \cdots P_1^n Q_1 Q_2 \cdots Q_m P_2^n \cdots P_2^1,$$

mapping to $\tilde{X}$ and bounding a disk diagram $D \to \tilde{X}$ with fixed carriers for the given hyperplanes. Suppose that P and D are chosen so that D has minimal complexity for all such

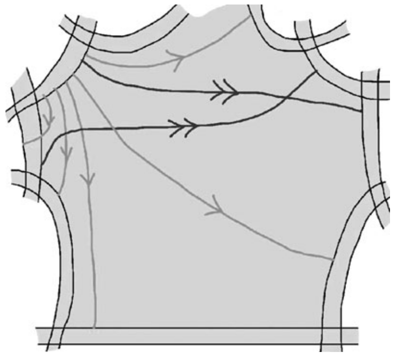

FIGURE 12. *The case $n = 3$. Double-arrowed dual curves are possible and single-arrowed ones lead to various contradictions.*

diagrams with those fixed carriers. Moreover, suppose that paths ρ and σ_i joining V^0, V_1^n, V_2^n are chosen in such a way that D has the minimal area among all such minimal complexity fixed-carrier diagrams constructed in this way; see Figure 12.

Observe that each path P_i^k for $1 \leqslant k \leqslant n-1$ has length at least 1, for otherwise $V_i^{k-1} \perp\!\!\!\!\!\perp V_i^{k+1}$, contradicting the grading. Similarly, each Q_j has length at least 1, for otherwise $U_{j-1} \perp\!\!\!\!\!\perp U_{j+1}$ (or, for example, $V_1^n \perp\!\!\!\!\!\perp U_2$ when $j = 1$), contradicting the minimum-length assumption on ρ.

Suppose that $n \geqslant 3$. Then $|P_0| = 0$. Indeed, if K is a dual curve emanating from P_0, then K cannot end on any Q_j, since K maps to a grade-1 hyperplane. Similarly, K cannot end on P_i^k for $k > 2$. If K ends on P_i^2, then we can replace D with a proper subdiagram by replacing V_i^1 by the hyperplane to which K maps. If K ends on P_i^1, then we apply Lemma 2.11 to produce a lower-complexity fixed-carrier diagram. If K has two ends on P_0, then P_0 is not a geodesic, a contradiction. Hence, K cannot exist, so $|P_0| = 0$ and $V_1^1 \perp\!\!\!\!\!\perp V_2^1$.

When $n = 3$, we thus have a path $V_1^3 \perp\!\!\!\!\!\perp V_1^2 \perp\!\!\!\!\!\perp V_1^1 \perp\!\!\!\!\!\perp V_2^1 \perp\!\!\!\!\!\perp V_2^2 \perp\!\!\!\!\!\perp V_2^3$ of length 5 joining V_1^3 to V_2^3. For $n \leqslant 3$, it is thus evident that $d_\Gamma(V_1^j, V_2^k) \leqslant 4$ when $1 \leqslant j < k \leqslant n$.

Now, with respect to V_1^1, the hyperplanes V_1^n and V_2^{n-1} have grade $n-1$ and lie in the same root based at V_1^1. By induction, there is a path $V_1^n \perp\!\!\!\!\!\perp W_1 \perp\!\!\!\!\!\perp \cdots \perp\!\!\!\!\!\perp W_d \perp\!\!\!\!\!\perp V_2^{n-1}$ with $d \leqslant 4$. Hence, for some W_p, the hyperplanes V_1^n and V_2^n lie in the full closed 3-ball about W_p and thus $d_\Gamma(V_1^n, V_2^n) \leqslant 5$, by applying the same argument to the full 3-ball centered at W_p. $\square$

In fact, the property of Γ stated in Lemma 4.3 characterizes graphs quasi-isometric to trees [**25**]. This fact was proved independently in [**14**].

5. *Weak hyperbolicity of cubulated groups and quasi-arboreal groups*

It follows from Theorem 4.1 that cocompactly cubulated groups satisfy a strong form of weak hyperbolicity, in which the coned-off Cayley graph with respect to hyperplane stabilizers is not only δ-hyperbolic but is actually a quasi-tree.

5.1. *Weak hyperbolicity and quasi-arboreality*

Farb defined a notion of relative hyperbolicity in terms of a *coned-off Cayley graph* in which the peripheral subgroups are associated to cone-points. While the additional property of *bounded coset penetration* is needed to actually obtain relative hyperbolicity in the sense of Gromov [**18**], the following notion of *weak hyperbolicity* is of interest.

Definition 5.1 (Weak hyperbolicity [**15**]). Let G be a finitely generated group and $\{G_W\}$ be a finite collection of subgroups. Let Γ be the graph obtained from the Cayley graph of G with respect to some finite generating set as follows. To the Cayley graph, add a vertex gG_W for each distinct coset of each G_W and join each gG_W by an edge to each vertex of the Cayley graph corresponding to an element of gG_W. The graph Γ is the *coned-off Cayley graph* of G relative to $\{G_W\}$. If there exists δ such that Γ is δ-hyperbolic, then G is *weakly hyperbolic relative to the collection* $\{G_W\}$.

Bowditch gave another definition, in which the coned-off Cayley graph is replaced by a G-graph with similar properties.

Definition 5.2 (Weak hyperbolicity [**4**]). Let G be a group and $\{G_W\}$ be a finite collection of subgroups. G is *weakly hyperbolic* relative to $\{G_W\}$ if G acts by isometries on a graph Γ with the following properties:

(1) The graph Γ is δ-hyperbolic for some δ:
(2) there are finitely many G-orbits of edges;
(3) each G_W fixes a vertex of Γ and each vertex stabilizer contains a conjugate of some G_W as a subgroup of finite index.

A G-graph Γ satisfying the latter two properties is a *generalized coset graph* for the pair $(G, \{G_W\})$, so that weak hyperbolicity amounts to the existence of a δ-hyperbolic generalized coset graph.

A stronger property is the following definition.

Definition 5.3 (Quasi-arboreal group). Let G be a group and $\{G_W\}$ be a finite collection of subgroups for which there is a generalized coset graph Γ such that Γ is quasi-isometric to a tree. Then G is *quasi-arboreal relative to the collection* $\{G_W\}$.

5.2. *Quasi-arboreality and cones on hyperplanes*

Let G be a finitely generated group acting on the CAT(0) cube complex $\tilde{X}$. Then G acts on the contact graph Γ by isometries, and the stabilizer of each vertex of Γ is exactly the stabilizer of the corresponding hyperplane.

The following discussion is, therefore, extraneous to the proof of Corollary 5.4, but gives a concrete viewpoint on the contact graph. Let $\tilde{X}$ be a CAT(0) cube complex with a set $\mathcal{W}$ of hyperplanes. The *coned-off complex* $\tilde{X}^*$ is obtained from $\tilde{X}$ by adding a cone on $N(W)$ for each $W \in \mathcal{W}$. More precisely,

$$\tilde{X}^* = \tilde{X} \sqcup \left(\coprod_{W \in \mathcal{W}} N(W) \times [-1, 1] \right) \big/ \{N(W) \times \{1\},\, N(W) \sim N(W) \times \{-1\}\}.$$

Associated to each hyperplane is a *cone-point*, which is joined by a *cone-edge* to each 0-cube in the corresponding hyperplane carrier.

The *coned-off hyperplane graph* is $C(\tilde{X}) = (\tilde{X}^*)^{(1)}$. When endowed with the combinatorial metric, $C(\tilde{X})$ is quasi-isometric to Γ, and to $\tilde{X}^*$ when $\tilde{X}$ is finite-dimensional.

Indeed, choose a map $\Gamma \to C(\tilde{X})$ that sends each vertex to the cone-point over the corresponding hyperplane. Each edge joins a pair of vertices corresponding to a pair of cone-points joined by a path in $C(\tilde{X})$ that is a concatenation of two cone-edges. Each edge of Γ maps linearly to some such length-2 path, giving a $(2, 0)$ quasi-isometric embedding $\Gamma \to C(\tilde{X})$. To see this, let U, V be hyperplanes, and let u, v be the corresponding cone-points of $C(\tilde{X})$. For any path $U = W_1 \perp\!\!\!\!\!- \cdots \perp\!\!\!\!\!- W_n = V$ in Γ joining U to V, there exists a path $P_1 P_2 \cdots P_{n-1}$ in $C(\tilde{X})$

joining u to v and having length $2(n-1)$. Indeed, P_i is a path consisting of two consecutive cone-edges, traveling from the cone-point associated to W_i to that associated to W_{i+1} via a 0-cube of $N(W_i) \cap N(W_{i+1})$, which exists by the definition of contacting hyperplanes. Hence, $d_{C(\tilde{X})}(u, v) \leqslant 2d_\Gamma(U, V)$.

On the other hand, let P be a geodesic path joining u to v, so that

$$P = K_1' B_1 K_2 B_2 \cdots K_n B_n K_{n+1}',$$

where each B_i is a (possibly trivial) path in $\tilde{X}^{(1)}$, each K_i is a concatenation of two cone-edges containing a single cone-point and K_1', K_{n+1}' are single cone-edges. Now, let c be a length-1 subpath of some B_i. Then there exists a path EF, consisting of cone-edges, with the same endpoints as c. Indeed, EF travels from the initial point of c, through the cone-point associated to the hyperplane dual to c, and ends at the terminus of c. Hence, there is a path of length $m = \sum_i |K_i| + 2\sum_i |B_i| + 2$ in $C(\tilde{X})$ that joins u and v and consists entirely of cone-edges; this path is obtained by replacing each 1-cube of each B_i by a length-2 cone-path in the preceding manner. This path has the form $E_1 F_1 \cdots E_k F_k$, where each E_i travels from a cone-point to a 0-cube of $\tilde{X}^{(1)}$, and each F_i travels from a 0-cube to a cone-point. Let U_i be the hyperplane corresponding to the cone-point that is the initial point of E_i (and terminal point of F_{i-1}). Then $U_1 \perp\!\!\!\!\!\!- U_2 \perp\!\!\!\!\!\!- \cdots \perp\!\!\!\!\!\!- U_{k+1}$ is a path in Γ joining U to V and having length $k = m/2$. Now, $|P| = \sum_i |K_i| + \sum_i |B_i| + 2 \geqslant m/2$, so that $d_{C(\tilde{X})}(u, v) \geqslant d_\Gamma(U, V)$.

Since every point of $\tilde{X}$ lies in some hyperplane carrier, every point of $C(\tilde{X})$ lies at distance at most $\frac{3}{2}$ from some cone-point, so that the map is quasi-surjective. Thus, $C(\tilde{X})$ is a quasi-tree by Theorem 4.1. That $\tilde{X}^*$ is quasi-isometric to $C(\tilde{X})$ when $\tilde{X}$ is finite-dimensional follows easily from the fact that, if $\dim \tilde{X} < \infty$, then $\tilde{X}$ and $\tilde{X}^{(1)}$ are quasi-isometric (see, for example, [**7**] for a proof).

Let G be a group acting properly and cocompactly on the CAT(0) cube complex $\tilde{X}$. Then G acts by isometries on $C(\tilde{X})$, with this action extending that of G on $\tilde{X}^{(1)}$. The stabilizer of a vertex of $C(\tilde{X})$ is finite when the vertex is a 0-cube of $\tilde{X}$ and equal to G_W, the stabilizer of the hyperplane W, for the vertex corresponding to W. The 1-cubes of $\tilde{X}$ have finite stabilizers by properness, and the cone-edges are finitely stabilized since they each have an initial vertex that is a 0-cube. Moreover, by cocompactness, there are finitely many orbits of edges.

Corollary 5.4. *Let G act on the* CAT(0) *cube complex $\tilde{X}$. Then G acts on a graph Γ that is quasi-isometric to a tree, such that the stabilizers of hyperplanes in $\tilde{X}$ correspond to the stabilizers of vertices in Γ.*

Furthermore, let $G \cong \pi_1 X$, with X a nonpositively curved cube complex with $\mathcal{W}$ the set of immersed hyperplanes in X. Suppose that $\mathcal{W}$ is finite and that there are finitely many contacts between immersed hyperplanes in X. (For instance, these hypotheses are satisfied when X is compact.) Then G is quasi-arboreal relative to the set $\{\pi_1 W : W \in \mathcal{W}\}$.

Example 5.5. The following groups act on quasi-trees by virtue of their actions on CAT(0) cube complexes.

(1) Finitely presented groups satisfying the $B(4) - T(4)$ small-cancelation condition act properly and cocompactly on CAT(0) cube complexes, and $B(6)$ groups act properly on CAT(0) cube complexes with finitely many orbits of hyperplanes [**35**].

(2) A right-angled Artin group R acts properly discontinuously and cocompactly on a CAT(0) cube complex that consists of Euclidean spaces of various dimensions, tiled by cubes, attached along affine subspaces [**10**]. The hyperplane stabilizers are themselves right-angled Artin groups.

(3) Farley proved that Thompson's group V acts properly discontinuously on a CAT(0) cube complex with two orbits of hyperplanes, one of which consists of trivially stabilized hyperplanes. Hence, V acts on a quasi-tree Γ. More generally, Farley gave an action on a CAT(0) cube complex for *diagram groups* associated to based semigroup presentations [**16**, **17**].

(4) Finitely generated Coxeter groups act properly on CAT(0) cube complexes with finitely many orbits of hyperplanes [**28**].

(5) Artin groups of type FC act on finite-dimensional CAT(0) cube complexes with 0-cube stabilizers of finite type [**9**].

The next example shows that there are noncubulated quasi-arboreal groups.

Example 5.6. Let $G \cong N \rtimes F$, where F is a finitely generated free group and N is a finitely generated group. Let Γ be the graph whose vertices are distinct cosets of N and whose edges correspond to left-multiplication by generators of $G/N \cong F$. Then G acts on Γ in such a way that the vertex-stabilizers are all N and the set of G-orbits of edges generates F. In fact, Γ is a Cayley graph for F and is thus a tree. The graph Γ is also a generalized coset graph showing that G is quasi-arboreal relative to N.

The subgroups N and F may be chosen in such a way that G does not act properly on a CAT(0) cube complex. For instance, the Baumslag-Solitar group with presentation $\langle a, b \mid (a^m)^b = a^n \rangle$ is quasi-arboreal relative to $\langle a \rangle$, since the resulting generalized coset graph is isometric to $\mathbb{R}$. However, a theorem of Haglund [**20**] states that G is not cubulated when $m \neq n$.

Definition 5.7. Let G be a finitely generated group and $\mathcal{G}$ be its Cayley graph with respect to some finite generating set. A subgroup $H \leqslant G$ is a *codimension-*1 *subgroup* if there exists $r \geqslant 0$ such that $\mathcal{G} - N_r(H)$ has two components, neither of which lies in $N_s(H)$ for any $s \geqslant 0$.

One verifies that, given an action of G on a CAT(0) cube complex, the hyperplane-stabilizers are codimension-1 subgroups. Conversely, Sageev's construction yields an action of G on a CAT(0) cube complex in the presence of a codimension-1 subgroup. A ready class of examples of groups without codimension-1 subgroups is that of groups having Kazhdan's Property (T) [**29**], and the following example shows that quasi-arboreality does not imply the existence of a codimension-1 subgroup.

Example 5.8. Consider the Steinberg presentation for $\mathrm{SL}_n(\mathbb{Z})$, with $n \geqslant 3$, where the generator a_{ij} represents the $n \times n$ matrix with diagonal entries equal to 1, the ij-entry equal to 1 and 0 elsewhere:

$$\mathrm{SL}_n(\mathbb{Z}) \cong \langle a_{ij}, 1 \leqslant i \neq j \leqslant n \mid [a_{ij}, a_{kl}],\ i \neq k, j \neq l;\ [a_{ij}, a_{jk}]a_{ik}^{-1},\ i \neq k;\ (a_{12}a_{21}a_{12}^{-1})^4 \rangle.$$

Let $A_{ij} = \langle a_{ij} \rangle$ and denote by Γ the coned-off Cayley graph of the pair $(\mathrm{SL}_n(\mathbb{Z}), \{A_{ij}\})$. A theorem of Carter and Keller implies that $\mathrm{SL}_n(\mathbb{Z})$ is boundedly generated with respect to $\{A_{ij}\}$ (see [**8**]). The graph Γ is therefore bounded, and hence $\mathrm{SL}_n(\mathbb{Z})$ is quasi-arboreal relative to $\{A_{ij}\}$. On the other hand, $\mathrm{SL}_n(\mathbb{Z})$ has Property (T) [**23**] and thus contains no codimension-1 subgroups.

6. *Asymptotic dimension*

6.1. *Asymptotic dimension of cube complexes*

In this section, we discuss the asymptotic dimension of groups acting on CAT(0) cube complexes and relate this to quasi-arboreality.

Definition 6.1 (Asymptotic dimension [1]). Let (M, d) be a metric space. The *asymptotic dimension* of M is at most n if for each $r > 0$ there exists a covering $M = \bigcup_{i \in I} U_i$ such that the sets U_i are uniformly bounded and no more than $n+1$ elements of $\{U_i\}_{i \in I}$ intersect any ball of radius r.

If $\operatorname{asdim} M \leqslant n$ and $\operatorname{asdim} M \nleqslant n-1$, then we say $\operatorname{asdim} M = n$. If no such n exists, then M is *asymptotically infinite-dimensional*.

The asymptotic dimension of a metric space is a quasi-isometry invariant and is thus well-defined for finitely generated groups. Word-hyperbolic groups have finite asymptotic dimension [19], but whether this is true of all CAT(0) groups is unknown.

Other examples of groups with finite asymptotic dimension are those that split as finite graphs of groups whose vertex-groups have finite asymptotic dimension [2] and groups that are hyperbolic relative to a finite collection of asymptotically finite-dimensional groups [31]. Theorem 6.2 states that a finite-dimensional CAT(0) cube complex is asymptotically finite-dimensional, and implies that any cubulated group is asymptotically finite-dimensional. More generally, Corollary 6.3 gives conditions under which the hypothesis of properness of the action can be relaxed. The following fundamental result was proved by Wright [37].

Theorem 6.2. *Let $\tilde{X}$ be a* CAT(0) *cube complex. Then* $\operatorname{asdim} \tilde{X} \leqslant \dim \tilde{X}$.

Wright also observes that a finitely generated group acting properly on a CAT(0) cube complex of dimension D has asymptotic dimension at most D. The main result of this section, Corollary 6.3, is a strengthening of Wright's result that also generalizes the theorem of Bell and Dranishnikov about graphs of asymptotically finite-dimensional groups.

Osin draws a striking contrast between relatively hyperbolic and weakly hyperbolic groups by giving examples of groups that are weakly hyperbolic relative to a finite collection of infinite cyclic subgroups but that contain free abelian groups of arbitrarily large rank and therefore have infinite asymptotic dimension. Osin's groups are also quasi-arboreal relative to that collection of cyclic subgroups: the coset graph is bounded [31]. On the other hand, these examples contain any recursively presentable group, and in particular have, for instance, subgroups with Property (T), and thus do not admit of proper essential actions on CAT(0) cube complexes, by an application of a result in [29].

We now prove that the hypothesis of properness of the action of G on $\tilde{X}$ can be relaxed in Wright's result; one requires only uniform boundedness of the asymptotic dimension of 0-cube stabilizers.

Corollary 6.3. *Let G be a finitely generated group acting on the locally finite* CAT(0) *cube complex $\tilde{X}$, with $\dim \tilde{X} = D < \infty$. Suppose there exists $n \in \mathbb{N}$ such that for each* 0*-cube x, the stabilizer G_x satisfies $\operatorname{asdim} G_x \leqslant n$. Then $\operatorname{asdim} G \leqslant n + D$.*

Proof. As usual, we will use the graph-metric on the 1-skeleton. In particular, if x is a 0-cube and $R \geqslant 0$, then $B_{\tilde{X}}(x, R)$ denotes the smallest subcomplex of $\tilde{X}$ containing all 0-cubes y with $d_{\tilde{X}}(x, y) \leqslant R$.

Let x_o be a 0-cube of $\tilde{X}$ and let $\psi : G \to \tilde{X}^{(1)}$ be $\psi(g) = gx_o$. This ψ is a Lipschitz map with respect to the word metric on G and the graph metric: the Lipschitz constant is $\max\{d_{\tilde{X}}(x_o, sx_o) : s \in \mathcal{S}\}$, where $\mathcal{S}$ is the finite generating set. Let $R \geqslant 0$ and let $x = gx_o$ for some $g \in G$. Then $B_{\tilde{X}}(x, R) = \{gy_1, gy_2, \ldots, y_b\}$, where $y_1, \ldots, y_b$ are the finitely many 0-cubes at distance at most R from x_o ($b < \infty$ since $\tilde{X}$ is locally finite). The preimage of $B_{\tilde{X}}(x, R)$ is therefore equal to $g(\bigcup_{i=1}^{b} \psi^{-1}(\{y_i\}))$. Now, if $y_i \notin Gx_o$, then $\psi^{-1}(\{y_i\}) = \emptyset$. Otherwise,

$y_i = g_i x_o$ for some $g_i \in G$, whence $\psi^{-1}(y_i) = g_i G_{x_o}$. Hence, the preimage of $B_{\tilde{X}}(x, R)$ is equal to $\bigcup_{i=1}^{b}(gg_i)G_{x_o}$, which is a finite union of cosets of G for which b depends on R but not on x. Therefore, $\{\psi^{-1}(B(gx_o, R)) \mid g \in G\}$ uniformly has asymptotic dimension bounded by $\operatorname{asdim} G_{x_o}$, which is at most n by hypothesis.

Therefore, by the Hurewicz-type theorem [**3**], $\operatorname{asdim} G \leqslant n + \operatorname{asdim} \psi(G)$. By Theorem 6.2, $\operatorname{asdim} \tilde{X} \leqslant D$. On the other hand, $\operatorname{asdim} \psi(G) \leqslant \operatorname{asdim} \tilde{X}$, and thus $\operatorname{asdim} G \leqslant n + D$, as required. □

7. *Hyperbolic cube complexes and complete bipartite subgraphs of* Γ

The aim of this section is to characterize non-δ-hyperbolic CAT(0) cube complexes in terms of the existence of certain complete bipartite subgraphs of their crossing- and contact graphs. This leads, in a sense, to a combinatorial version of the 'flat plane theorem' for cubulated groups. Similar results are proved in [**13**], from the point of view of median spaces.

Throughout this discussion, $\tilde{X}$ is a CAT(0) cube complex with contact graph Γ and crossing graph Δ.

7.1. *Flat plane theorem*

Definition 7.1 (Thin bicliques). The graph Γ has *thin bicliques* if there exists $n \in \mathbb{N}$ such that any complete bipartite subgraph $K_{p,q} \subseteq \Gamma$ satisfies $p < n$ or $q < n$.

The primary result is Theorem 7.3. We use the following version of the axiom of choice.

Lemma 7.2 (König's lemma). *Let Λ be a locally finite connected graph with infinitely many vertices and let R be a subdivided ray. Then for each vertex v, there is an embedding $R \hookrightarrow \Lambda$ containing v.*

Theorem 7.3. *Let G be a group acting properly and cocompactly on the* CAT(0) *cube complex $\tilde{X}$:*

(1) *G is word-hyperbolic if and only if the crossing graph Δ has thin bicliques*;
(2) *if G is not word-hyperbolic, then Δ contains the complete bipartite graph $K_{\infty,\infty}$.*

We postpone the proof of Theorem 7.3 until after that of Theorem 7.6, on which it depends, and also note that since $\Delta \subset \Gamma$, the complex $\tilde{X}$ is hyperbolic if Γ has thin bicliques.

7.2. *Hyperbolic* CAT(0) *cube complexes*

As usual, the graph $\tilde{X}^{(1)}$, with metric $d_{\tilde{X}}$, is *δ-hyperbolic* if for every geodesic triangle $\alpha_1\alpha_2\alpha_3 \to \tilde{X}^{(1)}$, each α_i lies in the δ-neighborhood of the union of the other two segments. The space $\tilde{X}$ with the CAT(0) piecewise-Euclidean metric is δ'-hyperbolic under the analogous condition on geodesic triangles.

The following lemma collects basic facts about hyperbolicity of cube complexes and the thin bicliques property of crossing graphs.

Lemma 7.4. *For a* CAT(0) *cube complex $\tilde{X}$ with crossing graph Δ, we have*:

(1) *if $\tilde{X}$ is finite-dimensional, then $\tilde{X}$, with the* CAT(0) *metric, is hyperbolic if and only if $\tilde{X}^{(1)}$ is a hyperbolic graph*;

(2) *if $\tilde{X}$ is infinite-dimensional, then it is not hyperbolic, and neither is* $\tilde{X}^{(1)}$;
(3) *if Δ has thin bicliques, then $\tilde{X}$ is finite-dimensional.*

Proof. (1) Follows from the fact that a finite-dimensional CAT(0) cube complex is quasi-isometric to its 1-skeleton.

To prove (2), note that for any $d \geqslant 0$, the existence of a d-cube guarantees the presence of a geodesic triangle, whose corners are 0-cubes, that is not d-thin. Hence, $\tilde{X}$ is not d-thin for any d if $\tilde{X}$ contains arbitrarily large cubes.

If Δ has thin bicliques, then there is an upper bound on the cardinality of cliques in Δ, since the existence of a complete subgraph on $2d$ vertices implies the existence of a complete (d, d) bipartite subgraph. The dimension of $\tilde{X}$ is the maximal cardinality of cliques in Δ, and (3) follows. □

When using disk diagrams, it is sometimes easier to think of a δ-hyperbolic space as one whose isoperimetric inequality is linear than it is to verify the thin triangle condition. Hence, we shall sometimes rely on the following version of Gromov's characterization of hyperbolic metric spaces, as those having linear isoperimetric inequality. This result is stated in cubical terms as follows.

LEMMA 7.5 [**18**]. *Let $\tilde{X}$ be a* CAT(0) *cube complex that is δ-hyperbolic with respect to its* CAT(0) *metric. There exists $\lambda \geqslant 0$ such that for each closed combinatorial path $\sigma \to \tilde{X}$, there exists a disk diagram $D \to \tilde{X}$ with $\partial_p D = \sigma$ such that the area of D is at most $\lambda|\sigma|$.*

Actually, only the fact that the isoperimetric function of a hyperbolic metric space is subquadratic is invoked in our applications.

7.3. *Complete bipartite subgraphs of Δ*

We first characterize hyperbolicity of the CAT(0) cube complex $\tilde{X}$ in terms of complete bipartite subgraphs of the crossing graph Δ. Recall that the *degree* of $\tilde{X}$ is the supremum over all 0-cubes $x \in \tilde{X}$ of the number of 1-cubes containing x. The main result of this subsection is the following theorem.

THEOREM 7.6. *The finite-degree* CAT(0) *cube complex $\tilde{X}$ is hyperbolic if and only if Δ has thin bicliques.*

Note that Theorem 7.6 implies that $\tilde{X}$ is hyperbolic when the contact-graph Γ has thin bicliques. Moreover, Lemma 7.14 does not require $\tilde{X}$ to have finite degree, so any cube complex whose contact graph has thin bicliques is hyperbolic. The proof of Theorem 7.6 is assembled as follows from the lemmas below.

Proof of Theorem 7.6. By Lemma 7.14, $\tilde{X}$ is hyperbolic when Δ has thin bicliques. Conversely, if Δ does not have thin bicliques, then by Lemma 7.9, $\tilde{X}$ does not have a linear isoperimetric function and thus, by Lemma 7.5, $\tilde{X}$ is not δ-hyperbolic for any δ. □

In the presence of a proper, cocompact group action on $\tilde{X}$, this yields the following.

Proof of Theorem 7.3. The group G is quasi-isometric to $\tilde{X}$. The first statement follows directly from Theorem 7.6, since $\tilde{X}$ has finite degree.

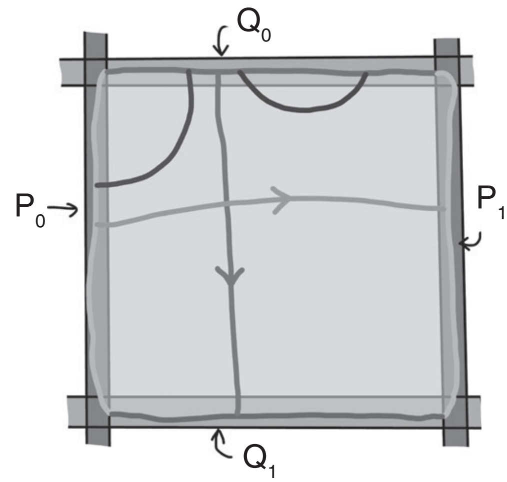

FIGURE 13. *The diagram arising from a* 4*-cycle in* Δ.

If G is not word-hyperbolic, then $\tilde{X}$ is not word-hyperbolic, and thus Δ contains $K_{n,n}$ for all $n \geqslant 0$, by Theorem 7.6. The proof of Lemma 7.9 shows that $\tilde{X}$, therefore, contains isometrically embedded copies of $[0,n]^2$ for any $n \geqslant 0$. Note that for each $n \geqslant 0$, there are only finitely many G-orbits of such $n \times n$-planar grids in $\tilde{X}$, because $\tilde{X}$ is locally finite and G acts cocompactly. Indeed, each orbit of $n \times n$ planar grid is represented by one of the finitely many grids of the given dimensions that intersect a fixed compact fundamental domain for the G-action.

Let Λ be the graph whose vertices correspond to G-orbits of isometrically embedded $n \times n$ planar grids in $\tilde{X}$, for all $n \geqslant 0$. The orbit represented by the $n \times n$ grid D_n is adjacent in Λ to the orbit represented by D_{n+1} if and only if $D_n \subset gD_{n+1}$ for some $g \in G$, and this adjacency describes all edges in Λ. Since $\tilde{X}$ contains D_n for all $n \geqslant 0$, the graph Λ is infinite. Furthermore, each vertex of Λ is joined by a path to one of the finitely many vertices corresponding to G-orbits of 0×0 planar grids (that is, 0-cubes). Hence, Λ has finitely many components, at least one of which must be infinite. Finally, Λ is locally finite since there are finitely many orbits of planar grid of each size, and each $n \times n$ planar grid is adjacent in Λ to vertices represented by $(n \pm 1) \times (n \pm 1)$ planar grids. Hence, Λ contains an infinite ray, by König's lemma, and thus there exists an infinite increasing union $D_0 \subset D_1 \subset \cdots$ of planar grids in $\tilde{X}$. Every hyperplane crossing D_n crosses D_m for $m > n$, and hence, we have an increasing union

$$K_{0,0} \subset K_{1,1} \subset \cdots \subset K_{n,n} \subset K_{n+1,n+1} \subset \cdots \subset \Delta$$

of crossing graphs of the planar grids, whence $K_{\infty,\infty} \subseteq \Delta$. $\square$

We now turn to the proof of Theorem 7.6.

DEFINITION 7.7 (Facing triple). The distinct hyperplanes H_1, H_2, H_3 form a *facing triple* if any two lie in a single halfspace associated to the third.

REMARK 7.8 (Planar grids from 4-cycles in Δ). Let $H_0 \bot V_0 \bot H_1 \bot V_1 \bot H_0$ be an embedded 4-cycle in Δ. For $i \in \{0,1\}$, choose concatenable geodesic paths $P_i \to N(V_i)$, $Q_i \to N(H_i)$ such that $A = P_0Q_0P_1Q_1$ is a closed path in $\tilde{X}$. Let $D \to \tilde{X}$ be a disk diagram with boundary path A. Suppose that A and D are chosen so that D has minimal complexity among all fixed carrier diagrams for the given 4-cycle in Δ; see Figure 13.

If K is a dual curve in D, then K travels from P_0 to P_1 or from Q_0 to Q_1. Indeed, K cannot travel from, say, P_1 to P_1, since P_1 is a geodesic. By Lemma 2.11, K cannot travel from P_i to Q_j, for then we could modify A, without affecting the 4-cycle in Δ, to produce a lower complexity fixed-carrier diagram. Similarly, no two dual curves emanating from P_i (or Q_i) can cross. Denote by $\mathbb{H}$ the set of dual curves traveling from P_0 to P_1 and by $\mathbb{V}$ the set of dual curves traveling from Q_0 to Q_1. Each element of $\mathbb{H}$ crosses each element of $\mathbb{V}$, and there are no other intersections of dual curves in D. Hence, D is a planar grid isomorphic to $P_0 \times Q_0$. In particular, D is a CAT(0) cube complex whose set of hyperplanes is $\mathbb{H} \sqcup \mathbb{V}$.

Let $H, H' \in \mathbb{H}$ and $V \in \mathbb{V}$ be dual curves in D. Since H crosses V, the dual curves H and V map to distinct hyperplanes of $\tilde{X}$, since hyperplanes in $\tilde{X}$ do not self-cross. Since H and H' are both dual to 1-cubes of P_0, and $P_0 \to \tilde{X}$ is a geodesic, the dual curves H, H' must map to distinct hyperplanes. Hence, the map $D \to \tilde{X}$ is injective on hyperplanes. Since $D \to \tilde{X}$ is a cubical map of CAT(0) cube complexes that is injective on hyperplanes, it is an isometric embedding.

Suppose $r \leqslant \min(|P_0|, |Q_0|)$. Then D contains an $r \times r$ planar grid E with boundary path P. Note that the map $D \to \tilde{X}$ restricts to an isometric embedding $E \to \tilde{X}$, and in particular, P embeds in $\tilde{X}$. Note that $|P| = 4r$, while $|E| = r^2$.

If there exists a disk diagram $F \to \tilde{X}$ with $\partial_p F = P$ and $\mathsf{Area}(F) < r^2$, then we could excise the interior of E from D and attach F along P to obtain a lower-area diagram D' with boundary path A, contradicting the fact that D has minimal area among diagrams with boundary path A. Hence, every disk diagram bounded by P has area at least r^2.

Lemma 7.9. *If $\tilde{X}$ has finite degree and Δ does not have thin bicliques, then $\tilde{X}$ is not δ-hyperbolic for any $\delta < \infty$.*

Proof. If the degree D of $\tilde{X}$ is 0, then $\tilde{X}$ is a 0-cube. If $D = 1$, then $\tilde{X}$ is a 1-cube. If $D = 2$, then either $\tilde{X}$ is a single 2-cube or $\tilde{X}$ is an interval. In each of these cases, Δ has thin bicliques and $\tilde{X}$ is hyperbolic.

If $D = 3$, then by Remark 7.8, Δ cannot contain an embedded 4-cycle and thus has thin bicliques. On the other hand, either $\tilde{X}$ is a single 3-cube, or $\tilde{X}$ embeds in $T \times [-\frac{1}{2}, \frac{1}{2}]$ for some tree T. Hence, $\tilde{X}$ is hyperbolic. Thus, we assume that $D > 3$.

For $2 \ll R < \infty$, let $\mathcal{H}, \mathcal{V}$ be disjoint sets of hyperplanes, with $\min(|\mathcal{H}|, |\mathcal{V}|) \geqslant R$, such that $K(\mathcal{V}, \mathcal{H}) \subseteq \Delta$, that is, for all $V \in \mathcal{V}$, $H \in \mathcal{H}$, we have $V \bot H$. Let V_0, V_1 be distinct hyperplanes in $\mathcal{V}$ and let H_0, H_1 be distinct hyperplanes in $\mathcal{H}$. Then $H_0 \bot V_0 \bot H_1 \bot V_1 \bot H_0$ is an embedded 4-cycle in Δ.

Without loss of generality, $\mathcal{V}$ and $\mathcal{H}$ are inseparable. Indeed, if W is a hyperplane separating $H, H' \in \mathcal{H}$, then W crosses each $V \in \mathcal{V}$, since $V \bot H$ and $V \bot H'$. Hence, we can include W in $\mathcal{H}$ without affecting the fact that $\mathcal{V}$ and $\mathcal{H}$ generate a biclique in Δ.

By Lemma 7.11, there exist isometrically embedded subcomplexes $A(\mathcal{V})$ and $A(\mathcal{H})$ such that the set of hyperplanes crossing $A(\mathcal{V})$ is exactly $\mathcal{V}$ and the set of hyperplanes crossing $A(\mathcal{H})$ is precisely $\mathcal{H}$. By the same lemma, $\tilde{X}$ contains an isometrically embedded subcomplex $A \cong A(\mathcal{V}) \times A(\mathcal{H})$.

By Lemma 7.12, for any $s \geqslant 0$, we can choose R large enough that $A(\mathcal{V})$ and $A(\mathcal{H})$, respectively, contain geodesic segments P, Q of length at least s. Hence, $A \subset \tilde{X}$ contains an $s \times s$ isometrically embedded planar grid E with $|\partial_p E| = 4s$. By Remark 7.8, and the fact that each of P and Q lies in a hyperplane-carrier, any disk diagram bounded by $\partial_p E$ has area at least $\mathsf{Area}(E) = s^2$. Thus, $\tilde{X}$ is not hyperbolic, by Lemmas 7.4 and 7.5. □

Remark 7.10. The lemma holds for $\tilde{X}$ of infinite maximal degree, under other interesting hypotheses. For example, if $\tilde{X}$ contains bicliques $K(\mathcal{V}, \mathcal{H})$ such that $\mathcal{V}$ and $\mathcal{H}$ are free of facing triples, and can be chosen arbitrarily large, then $\tilde{X}$ is not hyperbolic, by an argument very similar to that used to prove Lemma 7.9.

LEMMA 7.11. *Let $\mathcal{H}$ be a finite, inseparable set of hyperplanes. Then there exists an isometrically embedded compact subcomplex $A(\mathcal{H}) \subseteq \tilde{X}$ such that the set of hyperplanes crossing $A(\mathcal{H})$ is exactly $\mathcal{H}$.*

Moreover, if $\mathcal{V}$ is another finite inseparable set, and for all $V \in \mathcal{V}, H \in \mathcal{H}$, the hyperplanes H and V cross, then there is an isometric embedding $A(\mathcal{H}) \times A(\mathcal{V}) \hookrightarrow \tilde{X}$.

Proof. Let $\mathcal{H} = \{H_1, \ldots, H_m\}$. To produce $A(\mathcal{H})$, we shall argue by induction on m. In the base case, $m = 1$, we choose $A(\mathcal{H})$ to be any 1-cube dual to H_1. Then $A(\mathcal{H})$ is obviously compact and convex (and therefore isometrically embedded), and the unique hyperplane crossing $A(\mathcal{H})$ is H_1.

Suppose that $m \geqslant 2$ and suppose that the above labeling of the elements of $\mathcal{H}$ has the property that $\{H_1, \ldots, H_{m-1}\}$ is inseparable. Our induction hypothesis is that for any CAT(0) cube complex $\tilde{Y}$ and any collection $\{W_1, \ldots, W_{m-1}\}$ of hyperplanes in $\tilde{Y}$, there exists a compact convex subcomplex $A \subseteq \tilde{Y}$ such that the set of hyperplanes crossing A is precisely $\{H_1, \ldots, H_{m-1}\}$. In particular, this is hypothesized for any convex subcomplex of $\tilde{X}$, regarded as a CAT(0) cube complex in its own right.

Applying this hypothesis to $\tilde{X}$ itself yields a compact, convex subcomplex $A_{m-1} \subset \tilde{X}$ such that the set of hyperplanes that cross A_{m-1} is exactly $\{H_1, \ldots, H_{m-1}\}$.

The case $N(H_m) \cap A_{m-1} \neq \emptyset$: First suppose that $N(H_m) \cap A_{m-1}$ contains a 0-cube x. Since $x \in N(H_m)$, there exists a 1-cube e dual to H_m such that $x \in e$. Let $A' = A_{m-1} \cup e$ and let $A(\mathcal{H})$ be the cubical convex hull of A'. More precisely, $A(\mathcal{H})$ is the subcomplex whose 0-skeleton is the intersection of all halfspaces of $\tilde{X}^{(0)}$ that contain A'. By definition, each hyperplane crossing $A(\mathcal{H})$ must cross A', and therefore belongs to $\mathcal{H}$. Conversely, each hyperplane of $\mathcal{H}$ crosses A', and therefore $A(\mathcal{H})$. Indeed, the induction hypothesis ensures that H_i crosses $A_{m-1} \subset A'$ for $1 \leqslant i \leqslant m-1$, and H_m crosses $e \subset A'$. Being convex, $A(\mathcal{H})$ is isometrically embedded, and, since finitely many hyperplanes cross $A(\mathcal{H})$, it is compact.

The general case: Consider the set $\mathcal{A}_{m-1}$ of all compact, convex subcomplexes of $\tilde{X}$ that are crossed by exactly the set $\{H_1, \ldots, H_{m-1}\}$ of hyperplanes. Among these, choose A_{m-1} as close as possible to $N(H_m)$. If $d_{\tilde{X}}(N(H_m), A_{m-1}) = 0$, then the construction of $A(\mathcal{H})$ is complete. Hence, suppose that some hyperplane U separates A_{m-1} from H_m. Necessarily, $U \notin \mathcal{H}$.

For all $j \in \{1, \ldots, H_{m-1}\}$, the hyperplane U cannot separate H_m from H_j, and H_m and A_{m-1} lie in distinct halfspaces associated to U, and $N(H_j) \cap A_{m-1} \neq \emptyset$. It follows that $H_j \perp U$ for $1 \leqslant j \leqslant m-1$.

Denote by U_1 the copy of U bounding $N(U) \cong U \times [-1, 1]$ on the side contained in the halfspace associated to U that contains H_m. Then U_1 is a CAT(0) cube complex whose hyperplanes have the form $V \cap U_1$, where V is a hyperplane of $\tilde{X}$ that crosses U. Moreover, the map $V \mapsto V \cap U_1$ is a bijection from the set of hyperplanes of $\tilde{X}$ that cross U to the set of hyperplanes of U_1. Since $H_j \perp U$ for each j, the cube complex U_1 contains a hyperplane $H_j \cap U_1$ for $1 \leqslant j \leqslant m-1$.

Since U_1 is convex in $\tilde{X}$, any two hyperplanes $H_i \cap U_1, H_j \cap U_1$ of U_1 are separated by a hyperplane $V \cap U_1$ of U_1 if and only if H_i, H_j are separated in $\tilde{X}$ by V. Hence, $\{H_1 \cap U_1, \ldots, H_{m-1} \cap U_1\}$ is a set of $m-1$ inseparable hyperplanes in U_1. Applying our induction hypothesis to U_1 shows that there exists a compact convex subcomplex $B_{m-1} \subset U_1$ such that the set of hyperplanes of U_1 that cross B_{m-1} is precisely $\{H_1 \cap U_1, \ldots, H_{m-1} \cap U_1\}$.

Now, the inclusion $U_1 \hookrightarrow \tilde{X}$ embeds B_{m-1} in $\tilde{X}$ as a compact subcomplex. Moreover, since B_{m-1} is convex in U_1, and U_1 is convex in $\tilde{X}$, it follows that B_{m-1} is convex in $\tilde{X}$.

By construction, each H_j crosses B_{m-1}. Conversely, suppose that some hyperplane V crosses B_{m-1}. Then, regarding B_{m-1} as a subcomplex of U_1, we see that $V \cap U_1 = H_j \cap U_1$ for some

j, and therefore that $V = H_j$. We have verified that $B_{m-1} \in \mathcal{A}_{m-1}$. Since U_1 contains B_{m-1} and lies in the same halfspace associated to U as does H_m, the subcomplexes $N(H_m)$ and B_{m-1} of $\tilde{X}$ are not separated by U. Thus, B_{m-1} is an element of $\mathcal{A}_{m-1}$ that is strictly closer than A_{m-1} to $N(H_m)$, contradicting our choice of A_{m-1}. Hence, we can always choose A_{m-1} so that $A_{m-1} \cap N(H_m) \neq \emptyset$ and argue as above to produce $A(\mathcal{H})$.

Products: Let $\mathcal{H}, \mathcal{V}$ be finite, inseparable sets of hyperplanes such that each $H \in \mathcal{H}$ crosses each $V \in \mathcal{V}$. Since $\mathcal{V} \cup \mathcal{H}$ is inseparable, the above argument yields a compact, convex subcomplex $A(\mathcal{V} \cup \mathcal{H})$ such that the set of hyperplanes crossing $A(\mathcal{V} \cup \mathcal{H})$ is precisely $V \cup \mathcal{H}$. Similarly, we have convex, compact subcomplexes $A(\mathcal{V}), A(\mathcal{H})$ such that the set of hyperplanes crossing $A(\mathcal{V})$ (respectively, $A(\mathcal{H})$) is exactly $\mathcal{V}$ (respectively, $\mathcal{H}$).

Let $b \in A(\mathcal{V} \cup \mathcal{H})$ be a 0-cube. (As usual, for a hyperplane U, we denote by $b(U)$ the associated halfspace containing b.) Then $A(\mathcal{V})$ contains a unique 0-cube b_v, called the *projection* of b to $A(\mathcal{V})$, such that $b_v(V) = b(V)$ for all $V \in \mathcal{V}$, and $A(\mathcal{H})$ contains a 0-cube b_h such that $b_h(H) = b(H)$ for all $H \in \mathcal{H}$. Indeed, to construct b_v, we choose a halfspace $b_v(U)$ for each hyperplane U as follows. First, $b_v(U) = b(U)$ when $U \in \mathcal{V}$. Second, if $U \notin \mathcal{V} \cup \mathcal{H}$ and U does not separate $A(\mathcal{V} \cup \mathcal{H})$ from $A(\mathcal{V})$, then $b_v(U) = b(U)$. Third, if $U \in \mathcal{H}$, or if U separates $A(\mathcal{V} \cup \mathcal{H})$ from $A(\mathcal{V})$, then U crosses each element of $\mathcal{V}$, and we can let $b_v(U)$ be the complement of $b(U)$ without affecting consistency. Since b_v differs from b on finitely many hyperplanes, this orientation is canonical, and since the set of hyperplanes U for which $b_v(U) \neq b(U)$ is precisely the set of hyperplanes separating $A(\mathcal{V} \cup \mathcal{H})$ from $A(\mathcal{V})$, we have $b_v \in A(\mathcal{V})$. The projection b_h is constructed analogously.

Consider the map $A(\mathcal{V} \cup \mathcal{H})^{(0)} \to A(\mathcal{V})^{(0)} \times A(\mathcal{H})^{(0)}$ defined by $b \mapsto (b_v, b_h)$. For any $(b'_v, b'_h) \in A(\mathcal{V})^{(0)} \times A(\mathcal{H})^{(0)}$, let b' be the 0-cube of $\tilde{X}$ that orients hyperplanes not in $\mathcal{V} \cup \mathcal{H}$ toward a fixed 0-cube $b_o \in A(\mathcal{V} \cup \mathcal{H})$, and orients $V \in \mathcal{V}$ toward b_o if and only if b'_v orients V toward the projection of b_o, and orients $H \in \mathcal{H}$ toward b_o if and only if b'_h orients H toward the projection of b_o. Since elements of $\mathcal{V}$ can be oriented independently of elements of $\mathcal{H}$, this orientation is consistent, so b' is a genuine 0-cube. By construction, b' maps to (b'_v, b'_h), whence the given map is surjective. Now, by construction, 0-cubes $b, b' \in A(\mathcal{V} \cup \mathcal{H})$ differ exactly on those hyperplanes in $\mathcal{V}$ on which their projections to $A(\mathcal{V})$ differ and on those hyperplanes in $\mathcal{H}$ on which their projections to $A(\mathcal{H})$ differ. Hence, the given map is an isometric embedding, and it follows that $A(\mathcal{V} \cup \mathcal{H}) \cong A(\mathcal{V}) \times A(\mathcal{H})$, as desired. ☐

Lemma 7.12. *Let $\tilde{X}$ be a $\mathrm{CAT}(0)$ cube complex of finite degree. Then for all $s \geqslant 0$, there exists R such that if $\mathcal{H}$ is a finite, inseparable set of hyperplanes with $|\mathcal{H}| \geqslant R$, then $A(\mathcal{H})$ contains a geodesic segment of length at least s.*

Proof. Let $\mathcal{H}$ be a finite, inseparable set of hyperplanes, with $|\mathcal{H}| = R$. Since $A(\mathcal{H})$ is crossed by $R < \infty$ hyperplanes and is convex in $\tilde{X}$, it is a compact $\mathrm{CAT}(0)$ cube complex. Fix a base 0-cube $a_0 \in A(\mathcal{H})$ and let $\{a_1, \ldots, a_p\}$ be the finite set of 0-cubes a_i such that no geodesic segment γ joining a_0 to a_i is properly contained in a geodesic segment γ' emanating from a_0 (that is, the a_i are the 0-cubes 'as far as possible' from a_0, for $i \geqslant 1$). For $1 \leqslant i \leqslant p$, let γ_i be a geodesic segment in $A(\mathcal{H})$ joining a_0 to a_i.

The set $\{\gamma_i\}_{i=1}^{p}$ can be chosen so that $\mathcal{T} = \bigcup_i \gamma_i$ is an embedded (though not necessarily isometrically embedded) tree in $A(\mathcal{H})^{(1)}$. Moreover, $\mathcal{T}$ contains at least one 1-cube dual to each hyperplane in $\mathcal{H}$.

We first show that $\mathcal{T}$ can be chosen to be a tree. For any initial choice of $\{\gamma_i\}$, $\mathcal{T}$ is connected since each γ_i contains a_0. Now, if $\mathcal{T}$ contains a cycle, then for some $i \neq j$, there exist $\gamma'_i \subset \gamma_i, \gamma'_j \subset \gamma_j$ such that $\gamma'_i(\gamma'_j)^{-1}$ is a closed path joining a_0 to a 0-cube $b \in \gamma_i \cap \gamma_j$. By removing the interior of γ'_i from γ_i, we can replace γ_i with a geodesic γ''_i joining a_0 to a_i and consisting of γ'_j followed by the subpath of γ_i joining b to a_i. Indeed, either γ''_j is a geodesic, or some hyperplane H crosses γ_j' and separates b from a_i. Thus, H separates a_i from b and b from a_0. Hence, a_0 and a_i lie in the same halfspace associated to H, that is, H does not separate a_0 from a_i. Such an H cannot cross a geodesic γ_i joining a_0 to a_i, a contradiction. Hence, γ_j'' is a geodesic.

Let $\mathcal{T}'$ be the union of the paths $\{\gamma_k\}_{k\neq i} \cup \{\gamma_i''\}$. The subspace $\mathcal{T}'$ is again a union of maximal geodesics in $A(\mathcal{H})$, with one geodesic joining a_0 to each a_i. But $\mathcal{T}' \subset \mathcal{T}$ was formed by breaking a cycle, and hence $\mathrm{rk}(\pi_1\mathcal{T}') < \mathrm{rk}(\pi_1\mathcal{T})$. Since $\mathcal{T}$ is finite, repeating this procedure finitely many times yields the desired tree, which we henceforth denote by $\mathcal{T}$.

Since each $H \in \mathcal{H}$ separates at least two 0-cubes of $A(\mathcal{H})$, it is obvious that $\mathcal{T}$ contains at least one 1-cube dual to each hyperplane, and hence a total of at least R 1-cubes.

Next, let D be the degree of $\tilde{X}$. We shall show that for some $i \leqslant p$, the path γ_i has length at least $S = \log_{D-2}[(D-3)(R+1)+1] - 1$. Now, although $\mathcal{T} \hookrightarrow \tilde{X}$ is not an isometric embedding, γ_i is a geodesic of $\tilde{X}$ since it is a geodesic of $A(\mathcal{H})$ and the latter is isometrically embedded in $\tilde{X}$. Hence, $A(\mathcal{H}) \subset \tilde{X}$ contains a geodesic of length at least S. Choosing $\mathcal{H}$ to have cardinality

$$R \geqslant \frac{\exp[(s+1)\log(D-2)] - 1}{D-3} - 1$$

ensures that $A(\mathcal{H})$ contains a $\tilde{X}$-geodesic of length at least s.

To conclude, consider $\mathcal{T}$ as a tree rooted at a_0. Let $S = \max_i |\gamma_i|$, so that S is the depth of $\mathcal{T}$. Note that the depth of $\mathcal{T}$ is realized by a geodesic γ_i of $\tilde{X}$, and recall that $\mathcal{T}$ has at least R edges (and $R+1$ vertices). Let d be the maximal degree of a vertex of $\mathcal{T}$. In any case, $d \leqslant D$. In Lemma 7.9, $d \leqslant D-1$, since each vertex of $\mathcal{T}$ is contained in $A(\mathcal{H}) \times A(\mathcal{V})$, and has at least one incident 1-cube dual to a hyperplane not in $\mathcal{H}$. We also assume $d > 2$, since otherwise $\mathcal{T}$ is a line segment with at least R edges, and we have $S \geqslant R$.

View $\mathcal{T}$ as a subtree of a regular d-valent tree $\mathcal{U}$ rooted at a_0 having at least R edges and the same depth as $\mathcal{T}$. A computation shows that

$$R \leqslant \frac{(d-1)^{S+1} - 1}{d-2} - 1,$$

from which it follows that $S \geqslant \log_{d-1}[(R+1)(d-2)+1] - 1$ as desired. □

Remark 7.13. While the planar grid arising in $\tilde{X}$ from the complete bipartite graph $K \subset \Delta$ is isometrically embedded (in the combinatorial sense), it may not be convex. Any distortion of P in $\tilde{X}$ reflects some failure of K to be a full subgraph of Δ, by Lemma 2.13.

Lemma 7.14. *If Δ has thin bicliques, then $\tilde{X}$ is hyperbolic.*

Proof. By Lemma 7.4, it suffices to show that $\tilde{X}^{(1)}$ is δ-hyperbolic for some δ.

Suppose to the contrary that for any $n \in \mathbb{N}$, there exists a combinatorial geodesic triangle $\chi_n\eta_n\nu_n \to \tilde{X}^{(1)}$ that is not n-thin. Let $D_n \to \tilde{X}$ be a disk diagram of minimal area with boundary path $\chi_n\eta_n\nu_n$. By assumption, there exists a point $x \in \chi_n$ such that $d_{\tilde{X}}(x, \eta_n \cup \nu_n) > n$. Let $\mathcal{V}$ be the set of hyperplanes separating x from ν_n and let $\mathcal{H}$ be the set of hyperplanes separating x from η_n. Let $\mathbb{V}$ be the set of dual curves in D_n that separate x from ν_n in D_n and let $\mathbb{H}$ be the set of dual curves in D_n separating x from η_n. The diagram D_n is shown in Figure 14.

We first show that the map $D_n \to \tilde{X}$ induces bijections $\mathbb{V} \to \mathcal{V}$ and $\mathbb{H} \to \mathcal{H}$ and deduce that $|\mathbb{V}|, |\mathbb{H}| \geqslant n$. A disk diagram argument then shows that each element of $\mathbb{V}$ crosses each element of $\mathbb{H}$ and thus that $K(\mathcal{V}, \mathcal{H})$ is a complete bipartite subgraph of Δ with $|\mathcal{V}|, |\mathcal{H}| \geqslant n$. Hence, the failure of $\tilde{X}$ to be hyperbolic implies that Δ does not have thin bicliques.

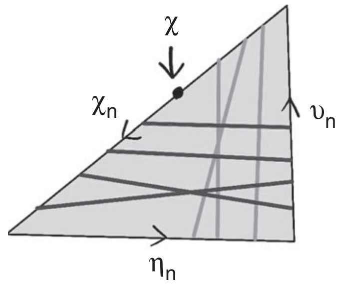

FIGURE 14. *The diagram D_n and some vertical and horizontal separating dual curves.*

The correspondences between $\mathbb{V}, \mathbb{H}$ *and* $\mathcal{V}, \mathcal{H}$: Dual curves in D_n map to distinct hyperplanes. Indeed, since each side of the triangle $\partial_p D_n$ is a geodesic segment, no dual curve has both endpoints on the same side, because a geodesic contains at most a single 1-cube dual to each hyperplane. Hence, if C, C' are distinct dual curves in D_n, then one of the sides χ_n, η_n, ν_n contains two of the four endpoints of $C \cup C'$. Thus, C and C' cannot map to the same hyperplane, for otherwise that side would cross a single hyperplane in two distinct 1-cubes, contradicting the fact that it is a geodesic. Hence, the maps $\mathbb{V}, \mathbb{H} \to \mathcal{V}, \mathcal{H}$ that associate dual curves in D_n to hyperplanes according to the map $D_n \to \tilde{X}$ are injective.

On the other hand, note that every element of $\mathbb{V}$ travels from χ_n to η_n. Indeed, no dual curve in D_n has both endpoints on the same side of the geodesic triangle. Hence, any $C \in \mathbb{V}$ travels from χ_n to η_n since it cannot cross ν_n and similarly any $C \in \mathbb{H}$ travels from χ_n to ν_n. Any geodesic joining x to some point of ν_n must cross each element of $\mathcal{V}$ exactly once, and thus each element of $\mathcal{V}$ occurs as a dual curve emanating from χ_n and terminating on η_n, that is, as an element of $\mathbb{V}$. The same argument holds for $\mathcal{H}$ and $\mathbb{H}$, and thus the desired correspondences between dual curves and hyperplanes are bijections.

Moreover, $|\mathcal{V}|, |\mathcal{H}| \geqslant n$, since the distance from x to η_n, ν_n is precisely the number of hyperplanes separating x from η_n, ν_n. Thus, $|\mathbb{V}|, |\mathbb{H}| \geqslant n$.

Crossing dual curves in D_n: Consider the decomposition $\chi_n = c_1 c_2 \cdots c_m$, where each c_i is a 1-cube, with c_1 initial and c_m terminal. Suppose $x \in c_p$. Then each element of $\mathcal{V}$ is dual to c_i with $i \leqslant p$ and each element of $\mathcal{H}$ is dual to c_i with $i \geqslant p$. The dual curve emanating from c_p belongs to $\mathbb{V}$, $\mathbb{H}$ or neither, according to the position of x on c_p. Since the elements of $\mathbb{V}$ end on η_n and the elements of $\mathbb{H}$ end on ν_n, each element of $\mathbb{V}$ crosses each element of $\mathbb{H}$, and hence, $\mathcal{V}$ and $\mathcal{H}$ are the two classes of a complete bipartite subgraph of Δ. $\square$

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